A semidiscrete scheme for a Penrose–Fife system and some Stefan problems in $\mathbb{R}^3$

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Abstract

This paper is concerned with the Penrose–Fife phase–field model and some Stefan problems, in which the heat flux is proportional to the gradient of the inverse absolute temperature. Recently, Colli and Sprekels proved that, as some parameters in the Penrose–Fife equations tend to zero, the corresponding solutions converge against the solutions to these Stefan problems.

Following their approach, we derive a time–discrete scheme for the Penrose–Fife equations, such that analogous convergence properties hold. Furthermore, we show some error estimates and prove the existence of solutions to the scheme.

1 Introduction

In [CS94], Colli and Sprekels considered an initial boundary value problem for a Penrose–Fife system with nonconserved order parameter, and proved that solutions to this problem for appropriate initial values converge against solutions to some Stefan problems, if the parameter δ or ε or both in the equation for the order parameter tend to zero.
In this paper, we introduce a time–discrete scheme for the Penrose–Fife system. We will repeat the a priori estimates derived in [CS94], and, following ideas from Colli, Horn, Laurençot, Sprekels, and Zheng (see [CS95, Lau94, HSZ93]), we derive some $L^\infty(\Omega)$–bounds. Using the same techniques as Colli in [Col95], we get error estimates for our scheme. Thus, we can prove convergence against the solution to the Penrose–Fife system, as the time step size tends to zero, and against the solution to one of the Stefan problems, as δ or ε or both also tend to zero. Moreover, we do not only derive an error estimate for the approximation of the relaxed–in–time Stefan problem by Penrose–Fife, similar to the one derived by Colli, but also for the other Stefan problems considered here.

2 Main results

2.1 Notations and desired problems

Before describing our problems, we introduce some auxiliary notations. Let β be the maximal monotone graph defined by

$$\beta(r) = \begin{cases} (-\infty, 0], & \text{for} \quad r = 0 \\ \{0\} \;, & \text{for} \quad 0 < r < 1 \\ [0, +\infty), & \text{for} \quad r = 1 \end{cases} \quad , \tag{2.1}$$

and let $(\cdot,\cdot)$ represent either the duality pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$ or the scalar product in $L^2(\Omega)$. Here, $\Omega \subset \mathbb{R}^3$ denotes a bounded domain with smooth boundary Γ.
We consider a problem with a Penrose–Fife system $(\mathbf{P}_{\delta\varepsilon})$, a relaxed–in–space Stefan problem $(\mathbf{P}_\varepsilon)$, a relaxed–in–time Stefan problem $(\mathbf{P}_\delta)$, and a Stefan problem $(\mathbf{P})$ as in [CS94].

$\boxed{(\mathbf{P}_{\delta\varepsilon}):}$ **Find a quadruple** (θ, u, χ, ξ) **fulfilling**

$$\theta \in H^1(0,T;L^2(\Omega)) \;\cap\; L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), \tag{2.2}$$

$$u \in H^1(0,T;L^2(\Omega)) \;\cap\; L^2(0,T;H^2(\Omega)) \cap L^\infty(\Omega_T), \tag{2.3}$$

$$\chi \in W^{1,\infty}(0,T;L^2(\Omega)) \;\cap\; H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega)), \tag{2.4}$$

$$\xi \in L^\infty(0,T;L^2(\Omega)), \tag{2.5}$$

$$\theta > 0, \qquad u = \frac{1}{\theta} \quad \text{a.e. in} \quad \Omega_T, \tag{2.6}$$

$$0 \le \chi \le 1, \qquad \xi \in \beta(\chi) \quad \text{a.e. in} \quad \Omega_T, \tag{2.7}$$

$$\begin{aligned}(\partial_t(c_0\theta + L\chi)(\cdot,t), v) \;&=\; \kappa \int_\Omega \nabla u(\cdot,t) \bullet \nabla v \,\mathrm{d}x + \int_\Gamma (\gamma u - \zeta)(\cdot,t) v \,\mathrm{d}\sigma \\ + (g(\cdot,t), v), \qquad & \forall\, v \in H^1(\Omega), \quad \text{for a.e.} \quad t \in (0,T),\end{aligned} \tag{2.8}$$

$$\delta\chi_t - \varepsilon\Delta\chi + \xi \;=\; L(u_C - u) \quad \text{a.e. in} \quad \Omega_T, \tag{2.9}$$

$$\frac{\partial\chi}{\partial n} \;=\; 0 \quad \text{a.e. in} \quad \Sigma, \tag{2.10}$$

$$\theta(\cdot,0) = \theta_s, \qquad \chi(\cdot,0) = \chi_s \quad \text{a.e. in} \quad \Omega. \tag{2.11}$$

$\boxed{(\mathbf{P}_{\delta}):}$ **Find a quadruple** (θ, u, χ, ξ) **fulfilling**

$$\theta \in L^\infty(0,T;L^2(\Omega)), \qquad u \in L^\infty(0,T;H^1(\Omega)), \quad \chi \in L^\infty(\Omega_T), \tag{2.12}$$

$$c_0\theta + L\chi \;\in\; W^{1,\infty}(0,T;H^1(\Omega)^*), \tag{2.13}$$

$$(c_0\theta + L\chi)(\cdot,0) \;=\; e_s \quad \text{in} \quad H^1(\Omega)^*, \tag{2.14}$$

(2.5)–(2.8), and

$$\chi \;\in\; W^{1,\infty}(0,T;L^2(\Omega)), \tag{2.15}$$

$$\delta\chi_t + \xi \;=\; L(u_C - u) \quad \text{a.e. in} \quad \Omega_T, \tag{2.16}$$

$$\chi(\cdot,0) \;=\; \chi_s \quad \text{a.e. in} \quad \Omega. \tag{2.17}$$

$\boxed{(\mathbf{P}_{\varepsilon}):}$ **Find a quadruple** (θ, u, χ, ξ) **fulfilling (2.12)–(2.14), (2.5)–(2.8), and**

$$\chi \;\in\; L^\infty(0,T;H^2(\Omega)), \tag{2.18}$$

$$-\varepsilon\Delta\chi + \xi \;=\; L(u_C - u) \quad \text{a.e. in} \quad \Omega_T, \tag{2.19}$$

$$\frac{\partial\chi}{\partial n} \;=\; 0 \quad \text{a.e. in} \quad \Sigma. \tag{2.20}$$

$\boxed{(\mathbf{P}):}$ **Find a quadruple** (θ, u, χ, ξ) **fulfilling (2.12)–(2.14), (2.5)–(2.8), and**

$$\xi = L(u_C - u) \quad \text{a.e. in} \quad \Omega_T. \tag{2.21}$$

Here, $T > 0$ stands for a final time, the positive constants c_0, κ, L, u_C represent physical constants, and we have set $\Omega_T := \Omega \times (0,T)$, $\Sigma := \Gamma \times (0,T)$.

2.2 Assumptions

It is required that

$$g \in L^\infty(\Omega_T)\,, \tag{2.22}$$

$$\gamma \in L^\infty(\Sigma), \qquad \gamma \geq c_\gamma \quad \text{a.e. in} \quad \Sigma, \quad \gamma_t \in L^\infty(\Sigma), \tag{2.23}$$

$$\zeta \in L^\infty(\Sigma), \qquad \zeta \geq c_\zeta \quad \text{a.e. in} \quad \Sigma, \quad \zeta_t \in L^\infty(\Sigma), \tag{2.24}$$

$$g_t \in L^2(0,T;L^\infty(\Omega)), \qquad \gamma \in L^\infty(0,T;C^1(\Gamma)), \quad \zeta \in L^\infty(0,T;H^{\frac{1}{2}}(\Gamma)), \tag{2.25}$$

hold for two positive constants c_γ, c_ζ. We assume that for the initial data θ_s, χ_s

$$e_s := c_0\theta_s + L\chi_s, \tag{2.26}$$

$$\theta_s \in H^1(\Omega), \qquad \theta_s > 0 \quad \text{a.e. in} \quad \Omega, \qquad \ln(\theta_s) \in L^\infty(\Omega), \tag{2.27}$$

$$\chi_s \in H^1(\Omega), \qquad 0 \leq \chi_s \leq 1 \quad \text{a.e. in} \quad \Omega, \tag{2.28}$$

$$u_s := \frac{1}{\theta_s} \in H^1(\Omega), \qquad a \leq u_s \leq b \quad \text{a.e. in} \quad \Omega, \tag{2.29}$$

hold for two positive constants a, b.
Except for the positive lower bound for ζ and the regularity assumption (2.25), these are the same assumptions as in Colli–Sprekels [CS94, (2.2)–(2.8)]. The lower bound for ζ is required to derive $L^\infty(\Omega)$–bounds for the approximation of θ, the regularity assumptions for γ and ζ are needed to prove the existence of an approximation for u in $H^2(\Omega)$. The regularity assumption for g_t is necessary for the error estimates.
From (2.27), we can obtain that (2.29) holds (see [CS94]).

2.3 The numerical scheme

Since in a numerical implementation one would like to change the time step size, we consider decompositions of $[0,T]$ that do not need to be uniform.

Definition 2.1 *An* admissible decomposition Z *is a finite subset Z of $[0,T]$ such that $Z = \{t_0, t_1, \ldots, t_K\}$ with $0 = t_0 < t_1 < \ldots < t_K = T$ and*

$$t_{m+1} - t_m \leq 2(t_m - t_{m-1}), \quad \forall\, 1 \leq m < K. \tag{2.30}$$

The width $|Z|$ *of the decomposition is defined by $|Z| := \max\limits_{1\leq m\leq K}(t_m - t_{m-1})$.*

Remark 2.1 In (2.30), the factor 2 could be replaced by any constant greater than 1. Since our decompositions will arise numerically by time step control, (2.30) is an upper bound for the new time step size.

For $\delta > 0, \varepsilon > 0$, $\theta_{0\delta\varepsilon}, u_{0\delta\varepsilon}, \chi_{0\delta\varepsilon} \in L^2(\Omega)$, and an admissible decomposition $Z = \{t_0, t_1, \ldots, t_K\}$, we define, for $1 \leq m \leq K$,

$$h_m := t_m - t_{m-1}, \qquad g_m(x) := \frac{1}{h_m}\int_{t_{m-1}}^{t_m} g(x,t)\,\mathrm{d}t\,, \qquad \forall x \in \Omega, \tag{2.31}$$

$$\gamma_m(\sigma) := \frac{1}{h_m}\int_{t_{m-1}}^{t_m} \gamma(\sigma,t)\,\mathrm{d}t\,, \qquad \zeta_m(\sigma) := \frac{1}{h_m}\int_{t_{m-1}}^{t_m} \zeta(\sigma,t)\,\mathrm{d}t\,, \quad \forall \sigma \in \Gamma, \tag{2.32}$$

as well as

$$e_{0\delta\varepsilon} = c_0\theta_{0\delta\varepsilon} + L\chi_{0\delta\varepsilon}, \tag{2.33}$$

and consider the problem:

$\boxed{(\mathbf{D}_{Z,\delta,\varepsilon})\text{:}}$ **For $1 \leq m \leq K$ find**

$$\theta_m \in L^2(\Omega), \qquad u_m, \chi_m \in H^2(\Omega), \qquad \xi_m \in L^2(\Omega) \tag{2.34}$$

such that

$$\begin{aligned}
\delta\frac{\chi_m - \chi_{m-1}}{h_m} - \varepsilon\Delta\chi_m + \xi_m &= L(u_C - u_m) \quad \text{a.e. in} \quad \Omega, && (2.35)\\
\xi_m &\in \beta(\chi_m) \quad \text{a.e. in} \quad \Omega, && (2.36)\\
\frac{\partial\chi_m}{\partial n} &= 0 \quad \text{a.e. in} \quad \Gamma, && (2.37)
\end{aligned}$$

and

$$\begin{aligned}
c_0\frac{\theta_m - \theta_{m-1}}{h_m} + L\frac{\chi_m - \chi_{m-1}}{h_m} + \kappa\Delta u_m &= g_m \quad \text{a.e. in} \quad \Omega, && (2.38)\\
0 < u_m, \quad \theta_m = \frac{1}{u_m} & \qquad \text{a.e. in} \quad \Omega, && (2.39)\\
-\kappa\frac{\partial u_m}{\partial n} = \gamma_m u_m - \zeta_m & \qquad \text{a.e. in} \quad \Gamma, && (2.40)
\end{aligned}$$

with

$$\theta_0 := \theta_{0\delta\varepsilon}, \qquad u_0 := u_{0\delta\varepsilon}, \qquad \chi_0 := \chi_{0\delta\varepsilon}. \tag{2.41}$$

Applying Green's formula, we can rewrite (2.38) and (2.40) as

$$\begin{aligned}
&\int\limits_\Omega \left(c_0\frac{\theta_m - \theta_{m-1}}{h_m} + L\frac{\chi_m - \chi_{m-1}}{h_m}\right) v\,\mathrm{d}x \,-\, \kappa\int\limits_\Omega \nabla u_m \bullet \nabla v\,\mathrm{d}x\\
&\qquad - \int\limits_\Gamma (\gamma_m u_m v - \zeta_m v)\,\mathrm{d}x \,=\, \int\limits_\Omega g_m v\,\mathrm{d}x\,, \quad \forall\, v \in H^1(\Omega).
\end{aligned} \tag{2.42}$$

We prove the following existence result in section 3:

Theorem 2.2 *Assume that (2.22)–(2.25) hold. For all $\delta > 0$, $\varepsilon > 0$, all admissible decompositions Z and all initial values $\theta_{0\delta\varepsilon}, u_{0\delta\varepsilon}, \chi_{0\delta\varepsilon} \in L^2(\Omega)$ there exists a unique solution to $(\boldsymbol{D}_{Z,\delta,\varepsilon})$.*

Remark 2.2 We use the solution to $(\mathbf{D}_{Z,\delta,\varepsilon})$ to construct an approximate solution $\left(\widehat{\theta}^{Z\delta\varepsilon}, \widehat{u}^{Z\delta\varepsilon}, \widehat{\chi}^{Z\delta\varepsilon}, \overline{\xi}^{Z\delta\varepsilon}\right)$ in $(L^\infty(0,T;L^2(\Omega)))^4$ to the Penrose–Fife system $(\mathbf{P}_{\delta\varepsilon})$. The function $\widehat{\theta}^{Z\delta\varepsilon}$ is defined linear in time on $[t_{m-1}, t_m]$, such that $\widehat{\theta}^{Z\delta\varepsilon}(t_k) = \theta_k$ holds for $k = 0, \ldots, K$. The functions $\widehat{u}^{Z\delta\varepsilon}$ and $\widehat{\chi}^{Z\delta\varepsilon}$ are defined analogously. We define $\overline{\xi}^{Z\delta\varepsilon}$ for $t \in (t_{m-1}, t_m]$ as $\overline{\xi}^{Z\delta\varepsilon}(t) = \xi_m$. We want to point out that neither $\widehat{\theta}^{Z\delta\varepsilon} = \frac{1}{\widehat{u}^{Z\delta\varepsilon}}$ nor $\overline{\xi}^{Z\delta\varepsilon} \in \beta\left(\widehat{\chi}^{Z\delta\varepsilon}\right)$ hold a.e. on Ω_T.

2.4 Convergence results and error estimates

We now state the main results of this paper. They are proved in sections 4 and 5.

Theorem 2.3 Approximation of the Penrose–Fife system:
Let $\delta > 0$, $\varepsilon > 0$ be fixed. Assume that (2.22)–(2.29),

$$\chi_s \in H^2(\Omega), \qquad \frac{\partial \chi_s}{\partial n} = 0 \quad a.e.\ in \quad \Gamma, \tag{2.43}$$

$$g_t \in L^\infty(\Omega_T), \tag{2.44}$$

$$\theta_{0\delta\varepsilon} = \theta_s, \qquad u_{0\delta\varepsilon} = u_s, \qquad \chi_{0\delta\varepsilon} = \chi_s, \tag{2.45}$$

hold. Let (θ, u, χ, ξ) be the solution to $(\boldsymbol{P}_{\delta\varepsilon})$. Then there is a positive constant C, independent of Z, such that

$$\left\|\widehat{\chi}^{Z\delta\varepsilon} - \chi\right\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} + \left\|\widehat{u}^{Z\delta\varepsilon} - u\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\widehat{\theta}^{Z\delta\varepsilon} - \theta\right\|_{L^2(0,T;L^2(\Omega))} \le C\sqrt{|Z|}, \tag{2.46}$$

and, as $|Z|$ tends to 0, it holds

$$\begin{aligned}
\widehat{\chi}^{Z\delta\varepsilon} \longrightarrow \chi \quad & \textit{weakly in} \quad H^1(0,T;H^1(\Omega)), && (2.47)\\
& \textit{weakly–star in} \quad W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)), && (2.48)\\
\overline{\xi}^{Z\delta\varepsilon} \longrightarrow \xi \quad & \textit{weakly–star in} \quad L^\infty(\Omega_T), && (2.49)\\
\widehat{u}^{Z\delta\varepsilon} \longrightarrow u \quad & \textit{weakly in} \quad H^1(0,T;L^2(\Omega)), && (2.50)\\
& \textit{weakly–star in} \quad L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T), && (2.51)\\
& \textit{weakly in} \quad L^2(t_*,T;H^2(\Omega))\ \forall\, 0 < t_* < T, && (2.52)\\
\widehat{\theta}^{Z\delta\varepsilon} \longrightarrow \theta \quad & \textit{weakly in} \quad H^1(0,T;L^2(\Omega)), && (2.53)\\
& \textit{weakly–star in} \quad L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T). && (2.54)
\end{aligned}$$

The existence of a unique solution to the Penrose–Fife equations follows from Proposition 2.6 in [CS94]. The assumption (2.44) is only needed since we use this Proposition. If this assumptions did not hold, it would still be possible to prove that the limit of the approximations is a solution to $(\mathbf{P}_{\delta\varepsilon})$.
The following Theorems correspond to the Theorems 2.8 to 2.10 in [CS94]. The existence and uniqueness of the solutions to the Stefan problems were proved by Colli and Sprekels in [CS94].

Theorem 2.4 Approximation of the relaxed–in–time Stefan problem:
Let $\delta > 0$ be fixed. Assume that (2.22)–(2.29), as well as

$$\theta_{0\delta\varepsilon} = \theta_s, \qquad u_{0\delta\varepsilon} = u_s, \tag{2.55}$$

$$\chi_{0\delta\varepsilon} - \varepsilon\Delta\chi_{0\delta\varepsilon} = \chi_s \quad a.e.\ in \quad \Omega, \tag{2.56}$$

$$\chi_{0\delta\varepsilon} \in H^2(\Omega), \qquad \frac{\partial \chi_{0\delta\varepsilon}}{\partial n} = 0 \quad a.e.\ in \quad \Gamma, \tag{2.57}$$

hold. Let (θ, u, χ, ξ) be the solution to the relaxed–in–time Stefan problem $(\boldsymbol{P}_\delta)$. For every $\bar{\varepsilon} > 0$ there is a positive constant C, independent of Z, such that for $0 < \varepsilon \le \bar{\varepsilon}$ it holds

$$\left\|\widehat{\chi}^{Z\delta\varepsilon} - \chi\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\widehat{u}^{Z\delta\varepsilon} - u\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\widehat{\theta}^{Z\delta\varepsilon} - \theta\right\|_{L^2(0,T;L^1(\Omega))} \le C\left(\sqrt{|Z|} + \sqrt{\varepsilon}\right). \tag{2.58}$$

As $|Z|$ and ε tend to 0, we have the convergences (2.49), (2.50), (2.51), and

$$\widehat{\chi}^{Z\delta\varepsilon} \longrightarrow \chi \quad \text{weakly–star in} \quad W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \tag{2.59}$$

$$\text{weakly–star in} \quad L^\infty(\Omega_T), \tag{2.60}$$

$$\widehat{\theta}^{Z\delta\varepsilon} \longrightarrow \theta \quad \text{weakly–star in} \quad L^\infty(0,T;L^2(\Omega)). \tag{2.61}$$

Theorem 2.5 Approximation of the relaxed–in–space Stefan problem:
Let $\varepsilon > 0$ be fixed. Assume that (2.22)–(2.29), (2.43), (2.45), and

$$-\varepsilon\Delta\chi_s + \beta(\chi_s) \ni L(u_C - u_s) \quad a.e.\ in \quad \Omega, \tag{2.62}$$

hold. Let (θ, u, χ, ξ) be the solution to the relaxed–in–space Stefan problem $(\boldsymbol{P}_\varepsilon)$. For every $\bar{\delta} > 0$ there is a positive constant C, independent of Z, such that for $0 < \delta \le \bar{\delta}$ it holds

$$\left\|\widehat{\chi}^{Z\delta\varepsilon} - \chi\right\|_{L^2(0,T;H^1(\Omega))} + \left\|\widehat{u}^{Z\delta\varepsilon} - u\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\widehat{\theta}^{Z\delta\varepsilon} - \theta\right\|_{L^2(0,T;L^2(\Omega))} \le C\left(\sqrt{|Z|} + \sqrt{\delta}\right). \tag{2.63}$$

As $|Z|$ and δ tend to 0, we have the convergences (2.49), (2.50), (2.51), (2.52), (2.53), (2.54), and

$$\widehat{\chi}^{Z\delta\varepsilon} \longrightarrow \chi \quad \text{weakly–star in} \quad H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega)). \tag{2.64}$$

Theorem 2.6 Approximation of the Stefan problem:
Assume that (2.22)–(2.29), (2.57),

$$\beta(\chi_s) \ni L(u_C - u_s) \quad a.e.\ in \quad \Omega, \tag{2.65}$$

$$u_{0\delta\varepsilon} - \frac{a}{2}\chi_{0\delta\varepsilon} = u_s - \frac{a}{2}\chi_s, \qquad \theta_{0\delta\varepsilon} = \frac{1}{u_{0\delta\varepsilon}} \quad a.e.\ in \quad \Omega, \tag{2.66}$$

$$-\varepsilon\Delta\chi_{0\delta\varepsilon} + \beta(\chi_{0\delta\varepsilon}) \ni L(u_C - u_{0\delta\varepsilon}) \quad a.e.\ in \quad \Omega, \tag{2.67}$$

hold. Let (θ, u, χ, ξ) be the solution to the Stefan problem $(\boldsymbol{P})$. For all $\bar{\delta} > 0$, $\bar{\varepsilon} > 0$ there is a positive constant C, independent of Z, such that for $0 < \delta \le \bar{\delta}$ and $0 < \varepsilon \le \bar{\varepsilon}$ it holds

$$\begin{aligned} &\left\|\overline{\xi}^{Z\delta\varepsilon} - \xi\right\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} + \left\|\widehat{u}^{Z\delta\varepsilon} - u\right\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} + \left\|\widehat{\theta}^{Z\delta\varepsilon} - \theta\right\|_{L^2(0,T;L^1(\Omega))} \\ \le\ & C\left(\sqrt{|Z|} + \delta^{\frac{1}{4}} + \varepsilon^{\frac{1}{4}}\right). \end{aligned} \tag{2.68}$$

As $|Z|$, δ, and ε tend to 0, we have the convergences (2.60), (2.61), and

$$\overline{\xi}^{Z\delta\varepsilon} \longrightarrow \xi \quad \text{weakly–star in} \quad L^\infty(0,T;L^2(\Omega)), \tag{2.69}$$

$$\widehat{u}^{Z\delta\varepsilon} \longrightarrow u \quad \text{weakly–star in} \quad H^1(0,T;L^{\frac{3}{2}}(\Omega)) \cap L^\infty(0,T;H^1(\Omega)). \tag{2.70}$$

Remark 2.3 If the assumption $g_t \in L^2(0,T;L^\infty(\Omega))$ in (2.25) is not fulfilled, the error estimates do not hold, but one can prove the convergences in the Theorems 2.3 to 2.6 using compactness arguments as in [CS94, Section 4].

Passing the error estimates for the approximation to the limit, we get an error estimate for the solution to the Penrose–Fife system with respect to the Stefan problems under consideration.

Corollary 2.7 *Assume that (2.22)–(2.29) and (2.44) hold.*

a) If $\theta_{0\delta\varepsilon}, u_{0\delta\varepsilon}$, and $\chi_{0\delta\varepsilon}$ are defined as in one of the Theorems 2.4 to 2.6, the problem $(\boldsymbol{P}_{\delta\varepsilon})$ with θ_s and χ_s replaced by $\theta_{0\delta\varepsilon}$ and $\chi_{0\delta\varepsilon}$, has a unique solution $\left(\theta^{\delta\varepsilon}, u^{\delta\varepsilon}, \chi^{\delta\varepsilon}, \xi^{\delta\varepsilon}\right)$.

b) Let $\delta > 0, \bar{\varepsilon} > 0$ be fixed and let (θ, u, χ, ξ) be the solution to the relaxed–in–time Stefan problem $(\boldsymbol{P}_\delta)$. If $\theta_{0\delta\varepsilon}, u_{0\delta\varepsilon}$, and $\chi_{0\delta\varepsilon}$ are defined as in Theorem 2.4, then there exists a positive constant C, such that, for $0 < \varepsilon \le \bar{\varepsilon}$,

$$\left\|\chi^{\delta\varepsilon} - \chi\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|u^{\delta\varepsilon} - u\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\theta^{\delta\varepsilon} - \theta\right\|_{L^2(0,T;L^1(\Omega))} \le C\sqrt{\varepsilon}. \quad (2.71)$$

c) Let $\bar{\delta} > 0, \varepsilon > 0$ be fixed and let (θ, u, χ, ξ) be the solution to the relaxed–in–space Stefan problem $(\boldsymbol{P}_\varepsilon)$. If $\theta_{0\delta\varepsilon}, u_{0\delta\varepsilon}$, and $\chi_{0\delta\varepsilon}$ are defined as in Theorem 2.5 and (2.43) as well as (2.62) hold, then there exists a positive constant C, such that, for $0 < \delta \le \bar{\delta}$,

$$\left\|\chi^{\delta\varepsilon} - \chi\right\|_{L^2(0,T;H^1(\Omega))} + \left\|u^{\delta\varepsilon} - u\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\theta^{\delta\varepsilon} - \theta\right\|_{L^2(0,T;L^2(\Omega))} \le C\sqrt{\delta}. \quad (2.72)$$

d) Let $\bar{\delta} > 0, \bar{\varepsilon} > 0$ be fixed and let (θ, u, χ, ξ) be the solution to the Stefan problem $(\boldsymbol{P})$. If $\theta_{0\delta\varepsilon}, u_{0\delta\varepsilon}$, and $\chi_{0\delta\varepsilon}$ are defined as in Theorem 2.6 and (2.65) holds, then there exists a positive constant C, such that, for $0 < \delta \le \bar{\delta}$ and $0 < \varepsilon \le \bar{\varepsilon}$,

$$\left\|\xi^{\delta\varepsilon} - \xi\right\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} + \left\|u^{\delta\varepsilon} - u\right\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} + \left\|\theta^{\delta\varepsilon} - \theta\right\|_{L^2(0,T;L^1(\Omega))} \le C\left(\delta^{\frac{1}{4}} + \varepsilon^{\frac{1}{4}}\right). \quad (2.73)$$

Proof. Assertion a) follows from [CS94, Proposition 2.6] and Lemma 4.1. For $K \in \mathbb{N}$ we define the admissible decomposition $Z = \{t_0, t_1, \ldots, t_K\}$ with $t_k := T\frac{k}{K}$. Theorem 2.3, Lemma 4.1, and a) yield, as K tends to ∞, that

$$\begin{aligned}
\widehat{\chi}^{Z\delta\varepsilon} &\longrightarrow \chi^{\delta\varepsilon} && \text{strongly in} \quad L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\\
\overline{\xi}^{Z\delta\varepsilon} &\longrightarrow \xi^{\delta\varepsilon} && \text{weakly–star in} \quad L^\infty(\Omega_T),\\
\widehat{u}^{Z\delta\varepsilon} \longrightarrow u^{\delta\varepsilon}, \quad \widehat{\theta}^{Z\delta\varepsilon} &\longrightarrow \theta^{\delta\varepsilon} && \text{strongly in} \quad L^2(0,T;L^2(\Omega)),
\end{aligned}$$

hold. Therefore, thanks to the weak–star lower semicontinuity of norms, (2.71), (2.72), resp. (2.73), follow from (2.58), (2.63), resp. (2.68). □

Remark 2.4 The error–estimate (2.71) is similar to the one derived by Colli in [Col95, Theorem 3], except for the norm for $\theta^{\delta\varepsilon} - \theta$. Colli estimates the $C^0([0,T], H^1(\Omega)^*)$–norm of this error.

3 Proof of Theorem 2.2

In this section, we will prove Theorem 2.2. First, we will have a close look at the approximation of the data. Next, we consider the equations for u_m and θ_m and finally the complete system (2.34)–(2.40).
In the sequel, we use the notation $\|\cdot\|$ for the $L^2(\Omega)$–norm and $\|\cdot\|_p$ for the $L^p(\Omega)$–norm, for all $p \in [1, \infty)$. $|\Omega|$ will denote the Lebesgue measure of the domain Ω.

3.1 Approximation of the data

Now we estimate the approximation of the data.

Lemma 3.1 *Assume that (2.22)–(2.25) hold. There exist positive constants C_a, C_b, C_c, independent of ε and δ, such that for all admissible decompositions $Z = \{t_0, t_1, \ldots, t_K\}$ it holds:*
The functions g_m, γ_m, and ζ_m defined in (2.31) and (2.32) fulfill, for $1 \le m \le K$,

$$\gamma_m \in C^1(\Gamma), \qquad \zeta_m \in H^{\frac{1}{2}}(\Gamma), \tag{3.1}$$

$$\|g_m\|_{L^\infty(\Omega)} + \|\gamma_m\|_{C^1(\Gamma)} + \|\zeta_m\|_{L^\infty(\Gamma)} + \|\zeta_m\|_{H^{\frac{1}{2}}(\Gamma)} \le C_a, \tag{3.2}$$

$$\gamma_m \ge c_\gamma, \qquad \zeta_m \ge c_\zeta \quad a.e.\ in \quad \Gamma, \tag{3.3}$$

$$\gamma_m v \in H^{\frac{1}{2}}(\Gamma), \qquad \|\gamma_m v\|_{H^{\frac{1}{2}}(\Gamma)} \le C_b \|v\|_{H^{\frac{1}{2}}(\Gamma)}, \quad \forall\, v \in H^{\frac{1}{2}}(\Gamma), \tag{3.4}$$

and, for $1 \le m < K$,

$$\left\| \frac{\gamma_{m+1} - \gamma_m}{h_m} \right\|_{L^\infty(\Gamma)} + \left\| \frac{\zeta_{m+1} - \zeta_m}{h_m} \right\|_{L^\infty(\Gamma)} \le C_c, \tag{3.5}$$

where the positive constants c_γ and c_ζ are specified in (2.23)–(2.24).

Proof. From (2.22)–(2.24) and (2.30) one can derive (3.2) elementary with $C_a := \|g\|_{L^\infty(\Omega_T)} + \|\gamma\|_{L^\infty(0,T;C^1(\Gamma))} + \|\zeta\|_{L^\infty(\Sigma)} + \|\zeta\|_{L^\infty(0,T;H^{\frac{1}{2}}(\Gamma))}$, the lower bounds in (3.3), and (3.5) with $C_b := 3\|\gamma_t\|_{L^\infty(\Sigma)} + 3\|\zeta_t\|_{L^\infty(\Sigma)}$. Since (2.25) yields that (3.1) holds, $v \mapsto \gamma_m v$ is a linear continuous mapping from both $L^2(\Gamma)$ and $H^1(\Gamma)$ into itself, with norm less than $\sqrt{3}\,\|\gamma_m\|_{C^1(\Gamma)}$. Since $H^{\frac{1}{2}}(\Gamma)$ is an interpolation space of $L^2(\Gamma)$ and $H^1(\Gamma)$ (see [Ama93, (5.2), (5.3), (5.19)]) it follows from (3.2) that (3.4) holds.

□

3.2 The temperature equation

First we consider the equations (2.38) and (2.40) for u_m.

Lemma 3.2 *Suppose that (2.22)–(2.25) hold. For all admissible decompositions $Z = \{t_0, t_1, \ldots, t_K\}$, all $m \in \{1, \ldots, K\}$, and all $f \in L^2(\Omega)$, there exists a unique solution $u \in H^2(\Omega)$ to*

$$-c_0 \frac{1}{h_m u} - \kappa \Delta u = f, \qquad u > 0 \quad a.e.\ in \quad \Omega, \tag{3.6}$$

$$\gamma_m u + \kappa \frac{\partial u}{\partial n} = \zeta_m \quad a.e.\ in \quad \Gamma. \tag{3.7}$$

To prove this Lemma, we will interpret (3.6), (3.7) as an operator equation and show that the corresponding operator is maximal monotone and surjective.

Lemma 3.3 *The operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ defined by*

$$Au = -\frac{c_o}{h_m u} - \kappa \Delta u \quad a.e.\ in \quad \Omega, \tag{3.8}$$

$$D(A) = \left\{ u \in H^2(\Omega) : u \quad fulfills\ (3.7) \quad and \quad u > 0 \quad a.e.\ in \quad \Omega \right\} \tag{3.9}$$

is maximal monotone.

Proof. Even if we cannot apply [Bré71, Corollary 13] directly, since γ_m and ζ_m are not constant in Γ, the proof of this corollary can be translated to our situation. Thus we will give only a sketch of the proof.

Using (3.3), we can, similarly to [Bré71, Theorem 12], derive a convex and lower semicontinuous mapping $\phi : L^2(\Omega) \to (-\infty, \infty]$, such that its subdifferential $\partial\phi$ fulfills

$$\partial\phi(u) = -\kappa\Delta u \quad \text{a.e. in} \quad \Omega, \quad D(\partial\phi) = \left\{ u \in H^2(\Omega) : u \quad \text{fulfills (3.7)} \right\}. \tag{3.10}$$

To prove that the operator defined in (3.10) is the subdifferential, we use that (3.1) and [Ama93, Theorem 9.2], for every $f \in L^2(\Omega)$, yield the existence of a unique solution to

$$u - \kappa\Delta u = f \quad \text{a.e. in} \quad \Omega, \quad -\kappa \frac{\partial u}{\partial n} = \gamma_m u - \zeta_m \quad \text{a.e. in} \quad \Gamma.$$

We define $k : \mathbb{R} \to (-\infty, \infty]$ by

$$k(x) = \begin{cases} -\frac{c_0}{h_m} \ln(x) & \text{for} \quad x > 0 \\ +\infty & \text{otherwise} \end{cases}. \tag{3.11}$$

This function is convex and lower semicontinuous. As in [Bré71, p. 115], we can derive a convex and lower semicontinuous mapping $\psi : L^2(\Omega) \to (-\infty, \infty]$, whose subdifferential fulfills $D(\partial\psi) = \left\{ u \in L^2(\Omega) : u > 0 \text{ a.e. in } \Omega, \quad \frac{1}{u} \in L^2(\Omega) \right\}$ and $v \in \partial\psi(u)$ if and only if $v = \frac{-c_0}{h_m u}$ a.e. in Ω. Analogously to [Bré71, Corollary 13], we obtain, using (3.3), (3.2) and [Bré71, Theorem 9], that $\partial\phi + \partial\psi = A$ is maximal monotone. $\square$

Lemma 3.4 *The operator A defined in Lemma 3.3 is coercive in $L^2(\Omega)$, i.e.*

$$\lim_{\|u\|\to\infty} \frac{(Au,u)}{\|u\|} = \infty. \tag{3.12}$$

Proof. Let u in $D(A)$ be arbitrary. Applying Green's formula, the definition of A, Young's inequality, and Lemma 3.1, we see that

$$\begin{aligned} (Au,u) &= \int_\Gamma (\gamma_m u - \zeta_m) u \,\mathrm{d}\sigma + \kappa \|\nabla u\|^2 - \frac{c_0}{h_m}|\Omega| \\ &\geq \frac{c_\gamma}{2}\|u\|^2_{L^2(\Gamma)} - \frac{1}{2c_\gamma}\int_\Gamma C_a^2 \,\mathrm{d}\sigma + \kappa\|\nabla u\|^2 - \frac{c_0}{h_m}|\Omega|. \end{aligned}$$

Recalling Lemma A.2, we infer that there are two positive constants C, C', such that $(Au,u) \geq C\|u\|^2_{H^1(\Omega)} - C'$ for all u in $D(A)$. Therefore (3.12) holds. □

Now we are going to prove Lemma 3.2. Let $f \in L^2(\Omega)$ be arbitrary. By Lemma 3.3 and Lemma 3.4, the operator A is maximal monotone and coercive. Therefore, recalling [Tib90, Chapter I, Theorem 2.4], we see that A is surjective. Thus we have $u \in D(A)$ with $Au = f$. Thanks to Lemma 3.3, u is a solution to (3.6) and (3.7).
Suppose we have another solution v. Rewriting (3.6) and (3.7) in terms of the differences, testing it by $u - v$, applying Green's formula and (3.3), we obtain

$$0 = \frac{c_0}{h_m}\int_\Omega \frac{(u-v)^2}{uv}\,\mathrm{d}x + \kappa\|\nabla(u-v)\|^2 + \int_\Gamma c_\gamma (u-v)^2 \,\mathrm{d}\sigma.$$

Since $uv > 0$ a.e. in Ω, this yields $u = v$. This finishes the proof. □

3.3 Existence of a solution to the system

Theorem 2.2 follows by induction and the following Lemma.

Lemma 3.5 *Assume that (2.22)–(2.25) are satisfied. Moreover, let any admissible decomposition $Z = \{t_0, t_1, \ldots, t_K\}$, any $m \in \{1, \ldots, K\}$, and any functions $\theta_{m-1}, \chi_{m-1} \in L^2(\Omega)$, be given. Then there exists a unique solution to (2.34)–(2.40).*

Proof. We will prove the existence of a solution to (2.34)–(2.40) via Schauder's fixed point theorem. It follows from [Bré71, Corollary 13] and Lemma 3.2 that for every $(u,\chi) \in L^2(\Omega) \times L^2(\Omega)$ there exists a unique solution $(\tilde{u}, \tilde{\chi}) =: \Psi(u,\chi)$ in $H^2(\Omega) \times H^2(\Omega)$ to

$$\delta\frac{\tilde{\chi}}{h_m} - \varepsilon\Delta\tilde{\chi} + \beta(\tilde{\chi}) \ni L(u_C - u) + \delta\frac{\chi_{m-1}}{h_m} \qquad \text{a.e. in } \Omega, \tag{3.13}$$

$$-c_0\frac{1}{h_m\tilde{u}} - \kappa\Delta\tilde{u} = -g_m + L\frac{\chi - \chi_{m-1}}{h_m} - c_0\frac{\theta_{m-1}}{h_m} \qquad \text{a.e. in } \Omega, \tag{3.14}$$

$$0 < \tilde{u} \qquad \text{a.e. in } \Omega, \tag{3.15}$$

$$\frac{\partial\tilde{\chi}}{\partial n} = 0, \quad \gamma_m\tilde{u} + \kappa\frac{\partial\tilde{u}}{\partial n} = \zeta_m \qquad \text{a.e. in } \Gamma. \tag{3.16}$$

This defines a mapping $\Psi : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$.
We define

$$\mathcal{M}_1 := \left\{ (u, \chi) \in H^1(\Omega) \times H^1(\Omega) : 0 \le \chi \le 1 \quad \text{a.e. in} \quad \Omega \right\}. \tag{3.17}$$

Let $(u, \chi) \in \mathcal{M}_1$ be arbitrary and define $(\tilde{u}, \tilde{\chi}) := \Psi(u, \chi)$. There are $\tilde{\xi}, \tilde{\theta} \in L^2(\Omega)$, such that

$$\delta \frac{\tilde{\chi} - \chi_{m-1}}{h_m} - \varepsilon \Delta \tilde{\chi} + \tilde{\xi} = L(u_C - u) \quad \text{a.e. in} \quad \Omega, \tag{3.18}$$

$$\tilde{\xi} \in \beta(\tilde{\chi}), \quad \tilde{\theta} = \frac{1}{\tilde{u}} \quad \text{a.e. in} \quad \Omega. \tag{3.19}$$

Obviously, by (3.14)—(3.16), any fixed point (u, χ) of Ψ yields a solution $\left(\tilde{\theta}, u, \chi, \tilde{\xi}\right)$ to (2.34)–(2.40), and vice versa. Therefore, it is sufficient to prove that Ψ has a unique fixed point.
We obtain from (3.13) that $\tilde{\chi}$ is in $D(\beta) = [0, 1]$ a.e. in Ω. Thus we have $(\tilde{u}, \tilde{\chi}) \in \mathcal{M}_1$.
We test (3.14) by $h_m \tilde{u}$, integrate, apply Green's formula, (3.16), (3.3) and Young's inequality, to obtain

$$\begin{aligned} -c_0 |\Omega| + h_m \kappa \|\nabla \tilde{u}\|^2 + h_m \int_\Gamma \left(\frac{c_\gamma}{2} \tilde{u}^2 - \frac{1}{2c_\gamma} \zeta_m^2 \right) \mathrm{d}\sigma \\ \le \int_\Omega \left(-h_m g_m + L(\chi - \chi_{m-1}) - c_0 \theta_{m-1} \right) \tilde{u} \, \mathrm{d}x \, . \end{aligned}$$

We conclude, using Lemma A.2, Schwarz's inequality, (3.2), and (3.17), that there is some constant $\tilde{C}_1 > 0$, such that

$$h_m \tilde{C}_1 \|\tilde{u}\|_{H^1(\Omega)}^2 \le c_0 |\Omega| + \frac{1}{2c_\gamma} \int_\Gamma C_a^2 \, \mathrm{d}\sigma + \left(h_m C_a \sqrt{|\Omega|} + L\sqrt{|\Omega|} + L \|\chi_{m-1}\| + c_0 \|\theta_{m-1}\| \right) \|\tilde{u}\| \, .$$

This yields, by Young's inequality, that $h_m \tilde{C}_1 \frac{1}{2} \|\tilde{u}\|_{H^1(\Omega)}^2 \le \tilde{C}_2$, with

$$\tilde{C}_2 := c_0 |\Omega| + \frac{1}{2c_\gamma} \int_\Gamma C_a^2 \, \mathrm{d}\sigma + \frac{1}{2\tilde{C}_1 h_m} \left((h_m C_a + L)\sqrt{|\Omega|} + L \|\chi_{m-1}\| + c_0 \|\theta_{m-1}\| \right)^2 .$$

For $\tilde{C}_3 := \frac{2}{h_m \tilde{C}_1} \tilde{C}_2$, we have $(\tilde{u}, \tilde{\chi})$ in $\mathcal{M}_2$ with

$$\mathcal{M}_2 := \left\{ (\bar{u}, \bar{\chi}) \in \mathcal{M}_1 : \|\bar{u}\|_{H^1(\Omega)}^2 \le \tilde{C}_3 \right\}. \tag{3.20}$$

In the sequel, we will assume $(u, \chi) \in \mathcal{M}_2$. Now, testing (3.18) with $h_m \tilde{\chi}$, applying Young's inequality, Green's formula, (3.16) and Schwarz's inequality, leads to

$$\frac{\delta}{2} \|\tilde{\chi}\|^2 - \frac{\delta}{2} \|\chi_{m-1}\|^2 + h_m \varepsilon \|\nabla \tilde{\chi}\|^2 + h_m \int_\Omega \tilde{\xi} \tilde{\chi} \, \mathrm{d}x \le h_m L \|u_C - u\| \, \|\tilde{\chi}\| \, .$$

Since we have $\tilde{\chi}$ in $[0, 1]$ and $\tilde{\xi}\tilde{\chi} \ge 0$ a.e. in Ω by (3.19) and the definition of β, this yields, by (3.20),

$$\frac{\delta}{2} \|\tilde{\chi}\|^2 + h_m \varepsilon \|\tilde{\chi}\|_{H^1(\Omega)}^2 \le \frac{\delta}{2} \|\chi_{m-1}\|^2 + h_m \varepsilon |\Omega| + h_m L \sqrt{|\Omega|} u_C + h_m L \sqrt{\tilde{C}_3} \sqrt{|\Omega|} =: \tilde{C}_4.$$

Thus, $(\tilde{u}, \tilde{\chi})$ is in $\mathcal{M}_3$ with

$$\mathcal{M}_3 := \left\{(\bar{u}, \bar{\chi}) \in \mathcal{M}_2 : h_m \varepsilon \left\|\bar{\chi}\right\|^2_{H^1(\Omega)} \leq \tilde{C}_4\right\}. \tag{3.21}$$

Since (3.21), (3.20), and (3.17) yield

$$\mathcal{M}_3 = \left\{(\bar{u}, \bar{\chi}) \in H^1(\Omega) \times H^1(\Omega) : h_m \varepsilon \left\|\bar{\chi}\right\|^2_{H^1(\Omega)} \leq \tilde{C}_4,\ \left\|\bar{u}\right\|^2_{H^1(\Omega)} \leq \tilde{C}_3,\ 0 \leq \bar{\chi} \leq 1 \text{ a.e. } \Omega\right\},$$

we obtain (see [Zei90b, (79c)]) that $\mathcal{M}_3$ is a nonempty, convex, compact set in $L^2(\Omega) \times L^2(\Omega)$ and, by construction, that Ψ maps $\mathcal{M}_3$ into itself. Since we obtain from Lemma 3.6 that Ψ is $(L^2(\Omega) \times L^2(\Omega))$–continuous, the Schauder fixed point theorem yields the existence of a fixed point of Ψ in $\mathcal{M}_3$.

Lemma 3.6 *Ψ is an $(L^2(\Omega) \times L^2(\Omega))$–continuous mapping.*

Proof. Let $(u^{(1)}, \chi^{(1)}), (u^{(2)}, \chi^{(2)}) \in L^2(\Omega) \times L^2(\Omega)$ be arbitrary, and

$$\left(\tilde{u}^{(i)}, \tilde{\chi}^{(i)}\right) := \Psi\left(u^{(i)}, \chi^{(i)}\right) \quad \text{for} \quad i = 1, 2,$$
$$u := u^{(1)} - u^{(2)}, \quad \chi := \chi^{(1)} - \chi^{(2)}, \qquad \tilde{u} := \tilde{u}^{(1)} - \tilde{u}^{(2)}, \quad \tilde{\chi} := \tilde{\chi}^{(1)} - \tilde{\chi}^{(2)}.$$

It follows from (3.13)–(3.16) that there are $\tilde{\xi}^{(1)}, \tilde{\xi}^{(2)} \in L^2(\Omega)$, such that

$$\delta \frac{\tilde{\chi}}{h_m} - \varepsilon \Delta \tilde{\chi} + \tilde{\xi}^{(1)} - \tilde{\xi}^{(2)} = -Lu \quad \text{a.e. in} \quad \Omega, \tag{3.22}$$
$$\tilde{u}^{(1)} > 0, \quad \tilde{u}^{(2)} > 0, \quad \tilde{\xi}^{(1)} \in \beta(\tilde{\chi}^{(1)}), \quad \tilde{\xi}^{(2)} \in \beta(\tilde{\chi}^{(2)}) \quad \text{a.e. in} \quad \Omega, \tag{3.23}$$
$$c_0 \frac{\tilde{u}}{h_m \tilde{u}^{(1)} \tilde{u}^{(2)}} - \kappa \Delta \tilde{u} = L \frac{\chi}{h_m} \quad \text{a.e. in} \quad \Omega, \tag{3.24}$$
$$\frac{\partial \tilde{\chi}}{\partial n} = 0, \quad \gamma_m \tilde{u} + \kappa \frac{\partial \tilde{u}}{\partial n} = 0 \quad \text{a.e. in} \quad \Gamma. \tag{3.25}$$

Testing (3.22) with $\tilde{\chi}$, applying Green's formula, (3.25), and the inequality $(\tilde{\xi}^{(1)} - \tilde{\xi}^{(2)})\tilde{\chi} \geq 0$ (by (3.23) and the monotonicity of β), we get

$$\frac{\delta}{h_m} \left\|\tilde{\chi}\right\|^2 + \varepsilon \left\|\nabla \tilde{\chi}\right\|^2 \leq -L \int_\Omega u \tilde{\chi} \, \mathrm{d}x. \tag{3.26}$$

Testing (3.24) with $h_m \tilde{u}$, using Green's formula, (3.25), (3.3), and Schwarz's inequality, we find

$$c_0 \int_\Omega \frac{\tilde{u}^2}{\tilde{u}^{(1)} \tilde{u}^{(2)}} \, \mathrm{d}x + h_m c_\gamma \left\|\tilde{u}\right\|^2_{L^2(\Gamma)} + h_m \kappa \left\|\nabla \tilde{u}\right\|^2 \leq L \int_\Omega \chi \tilde{u} \, \mathrm{d}x.$$

Using Lemma A.2, (3.23) and Young's inequality, we get for some $\tilde{C}_5 > 0$,

$$h_m \tilde{C}_5 \left\|\tilde{u}\right\|^2_{H^1(\Omega)} \leq L \int_\Omega \chi \tilde{u} \, \mathrm{d}x. \tag{3.27}$$

Adding (3.26) and (3.27), and applying Young's inequality, we find

$$\frac{\delta}{h_m}\|\tilde{\chi}\|^2 + \varepsilon\|\nabla\tilde{\chi}\|^2 + h_m\tilde{C}_5\|\tilde{u}\|^2_{H^1(\Omega)} \le -L\int_\Omega u\tilde{\chi}\,\mathrm{d}x + L\int_\Omega \chi\tilde{u}\,\mathrm{d}x \tag{3.28}$$

$$\le \frac{L}{2\delta}h_m\|u\|^2 + \frac{\delta}{2h_m}\|\tilde{\chi}\|^2 + \frac{1}{2\tilde{C}_5 h_m}L^2\|\chi\|^2 + h_m\frac{\tilde{C}_5}{2}\|\tilde{u}\|^2_{H^1(\Omega)}.$$

Thus, we have proved that Ψ is Lipschitz–continuous on $L^2(\Omega)\times L^2(\Omega)$. □

It remains to show the uniqueness of the fixed point. Suppose $(u^{(1)},\chi^{(1)})$, $(u^{(2)},\chi^{(2)})$ are two fixed points of Ψ. Thus we can repeat the estimates in the proof of Lemma (3.6) with $\tilde{u}=u$ and $\tilde{\chi}=\chi$. Hence, (3.28) yields $u^{(1)}=u^{(2)}$ and $\chi^{(1)}=\chi^{(2)}$ in $H^1(\Omega)$.

□

4 Uniform Estimates

In the sequel, we will assume that there are two positive upper bounds $\overline{\delta}$ and $\overline{\varepsilon}$ for δ and ε, i.e.

$$0<\delta\le\overline{\delta}, \qquad 0<\varepsilon\le\overline{\varepsilon}. \tag{4.1}$$

We will consider initial values $\theta_{0\delta\varepsilon}, u_{0\delta\varepsilon}, \chi_{0\delta\varepsilon}$ and assumptions corresponding to one of the Theorems 2.3 to 2.6, $\delta>0$, $\varepsilon>0$ with (4.1), and an admissible decomposition $Z=\{t_0,t_1,\dots,t_K\}$.

In the sequel, C_i, for $i\in\mathbb{N}$, will always denote positive generic constants, independent of ε and the decomposition Z. They may depend on δ, if $(\mathbf{P}_{\delta\varepsilon})$ or $(\mathbf{P}_{\delta})$ is considered as limit problem, but they are independent of δ, if we consider $(\mathbf{P}_{\varepsilon})$ or $(\mathbf{P})$ as limit problem. Thus the constants only depend on δ if δ is fixed.

We start by deriving some properties of the initial values. The first Lemma modifies Lemma 3.2 in [CS94].

Lemma 4.1 *The initial values $\theta_{0\delta\varepsilon}, u_{0\delta\varepsilon}$, and $\chi_{0\delta\varepsilon}$ considered in the statements of Theorems 2.3 to 2.6 are uniquely determined and satisfy (2.57). Moreover, if we define $\xi_{0\delta\varepsilon}\in L^2(\Omega)$ by $\xi_{0\delta\varepsilon}=0$ if $(\boldsymbol{P}_{\delta\varepsilon})$ or $(\boldsymbol{P}_{\delta})$ is concerned (see Theorems 2.3 and 2.4), and by $\xi_{0\delta\varepsilon}=\varepsilon\Delta\chi_{0\delta\varepsilon}+L(u_C-u_{0\delta\varepsilon})$ otherwise (see Theorems 2.5 and 2.6), and if we define $\chi_{-1}\in L^2(\Omega)$ by*

$$\delta\frac{\chi_{0\delta\varepsilon}-\chi_{-1}}{h_0} - \varepsilon\Delta\chi_{0\delta\varepsilon} + \xi_{0\delta\varepsilon} = L(u_C-u_{0\delta\varepsilon}) \quad a.e.\ in \quad \Omega, \tag{4.2}$$

with $h_0:=|Z|$, then (2.35)–(2.37) hold for $m=0$, and there are positive constants C_1, C_2, and C_3, such that

$$\|u_{0\delta\varepsilon}\|_{H^1(\Omega)}\le C_1, \quad \frac{a}{2}\le u_{0\delta\varepsilon}\le b+\frac{a}{2}, \qquad \theta_{0\delta\varepsilon}=\frac{1}{u_{0\delta\varepsilon}} \quad a.e.\ in \quad \Omega, \tag{4.3}$$

$$\varepsilon\|\chi_{0\delta\varepsilon}\|^2_{H^2(\Omega)}+\|\chi_{0\delta\varepsilon}\|^2_{H^1(\Omega)}+\|e_{0\delta\varepsilon}\|^2 \;\le\; C_2, \tag{4.4}$$

$$\xi_{0\delta\varepsilon}\in\beta(\chi_{0\delta\varepsilon}) \qquad a.e.\ in \quad \Omega, \tag{4.5}$$

$$\delta\left\|\frac{\chi_{0\delta\varepsilon}-\chi_{-1}}{|Z|}\right\|^2 \;\le\; C_3, \tag{4.6}$$

with the constants a, b being specified in (2.29). In the situation of Theorems 2.4 and 2.6, there is a positive constant C_4, such that

$$\|\chi_{0\delta\varepsilon} - \chi_s\|^2 + \|e_{0\delta\varepsilon} - e_s\|^2 \le \varepsilon C_4. \tag{4.7}$$

Proof. First we examine the situation of Theorem 2.3. Owing to (2.45), $\theta_{0\delta\varepsilon}$, $u_{0\delta\varepsilon}$, and $\chi_{0\delta\varepsilon}$ are uniquely determined as θ_s, u_s, and χ_s. Using the assertions (2.26), (2.28), (2.29), and (2.43) for the initial data as well as (2.33), (4.2) and (4.1), we obtain the assertions of the lemma.
In the framework of Theorems 2.4 to 2.6, $\theta_{0\delta\varepsilon} = \frac{1}{u_{0\delta\varepsilon}}$ follows from (2.29) and (2.55), resp. (2.45), resp. (2.66). For the remaining assertions, except (4.4) and (4.7), we refer to [CS94, Lemma 3.2]. From the estimates in the proof of [CS94, Lemma 3.2] we obtain (see [CS94, (3.11) resp. (3.15)]) a uniform bound for $\|\nabla\chi_{0\delta\varepsilon}\|^2 + \varepsilon\,\|\Delta\chi_{0\delta\varepsilon}\|^2$. From (4.5) we obtain $0 \le \chi_{0\delta\varepsilon} \le 1$ a.e. on Ω. Therefore, we have a uniform bound for $\|\chi_{0\delta\varepsilon}\|^2_{H^1(\Omega)}$ and, by (2.57) and Lemma A.3, also one for $\varepsilon\,\|\chi_m\|^2_{H^2(\Omega)}$. From (2.33), (4.3), and (4.5), it follows that $\|e_{0\delta\varepsilon}\|^2 \le (\frac{2c_0}{a} + L)\,|\Omega|$. Therefore (4.4) holds.
In the situation of Theorem 2.4, using (2.33), (2.26), (2.56), (2.55), and (4.4), we get

$$\|\chi_{0\delta\varepsilon} - \chi_s\|^2 + \|e_{0\delta\varepsilon} - e_s\|^2 = (1 + L^2)\varepsilon^2\,\|\Delta\chi_{0\delta\varepsilon}\|^2 \le (1 + L^2)\varepsilon C_2.$$

Finally, we examine the situation in Theorem 2.6. Testing (2.66) with $u_{0\delta\varepsilon} - u_s$ and $\chi_{0\delta\varepsilon} - \chi_s$, we see that

$$\frac{2}{a}\,\|u_{0\delta\varepsilon} - u_s\|^2 = \int_\Omega (\chi_{0\delta\varepsilon} - \chi_s)\,(u_{0\delta\varepsilon} - u_s)\;dx\; = \frac{a}{2}\,\|\chi_{0\delta\varepsilon} - \chi_s\|^2. \tag{4.8}$$

Using (2.65), (2.67), the monotonicity of β, Green's formula, (2.57), and Young's inequality, we obtain

$$0 \le L\int_\Omega (u_{0\delta\varepsilon} - u_s)\,(\chi_s - \chi_{0\delta\varepsilon})\;dx\; + \varepsilon\left(\frac{1}{2}\,\|\nabla\chi_s\|^2 - \frac{1}{2}\,\|\nabla\chi_{0\delta\varepsilon}\|^2\right).$$

Therefore, we conclude by (4.8), (2.28), (2.33), (2.26), (2.29), and (4.3), that (4.7) holds. □

Since (4.3) and (2.57) yield that $\theta_{0\delta\varepsilon}, u_{0\delta\varepsilon}, \chi_{0\delta\varepsilon}$ are in $L^2(\Omega)$, it follows from Theorem 2.2 that $(\mathbf{D}_{Z,\delta,\varepsilon})$ has a unique solution $\theta_0, u_0, \chi_0,\ \theta_1, u_1, \chi_1, \xi_1, \ldots, \theta_K, u_K, \chi_K, \xi_K$. We are going to derive some uniform estimates for this solution. The generic constants will also be independent of the solution.

Remark 4.1 a) In the framework of Theorem 2.3 and Theorem 2.5, we obtain from (2.45), (2.26), and (2.33) that $\chi_{0\delta\varepsilon} = \chi_s$ and $e_{0\delta\varepsilon} = e_s$ hold. Since ε tends to 0 and (4.7) holds in the situation of Theorem 2.4 and Theorem 2.6, we conclude that for the limits considered in one of the Theorems 2.3 to 2.6, we have

$$\|\chi_{0\delta\varepsilon} - \chi_s\|^2 + \|e_{0\delta\varepsilon} - e_s\|^2 \longrightarrow 0. \tag{4.9}$$

b) We obtain from (2.39), (2.36), (2.41), (4.3), and (4.5) that

$$0 < u_m, \quad \theta_m = \frac{1}{u_m}, \quad 0 \le \chi_m \le 1 \quad \text{a.e. in} \quad \Omega, \quad \forall\, 0 \le m \le K. \tag{4.10}$$

The following seven lemmas correspond to Lemmas 3.3 to 3.10 in [CS94], except for the estimate for the $L^\infty(\Omega)$–Norm of ξ_m in (4.12). First we work on (2.35).

Lemma 4.2 *For all $1 \le k \le K$ it holds*

$$\begin{aligned} &\frac{\delta}{2}\left\|\frac{\chi_k - \chi_{k-1}}{h_k}\right\|^2 + \varepsilon \sum_{m=1}^{k} h_m \left\|\nabla\left(\frac{\chi_m - \chi_{m-1}}{h_m}\right)\right\|^2 \\ &\quad \le \frac{C_3}{2} - \sum_{m=1}^{k} h_m \int_\Omega L \frac{u_m - u_{m-1}}{h_m}\frac{\chi_m - \chi_{m-1}}{h_m}\,\mathrm{d}x\,, \end{aligned} \tag{4.11}$$

$$\|\xi_k\| \le \|L(u_C - u_k)\|\,, \qquad \|\xi_k\|_{L^\infty(\Omega)} \le \|L(u_C - u_k)\|_{L^\infty(\Omega)}\,, \tag{4.12}$$

$$\frac{\delta}{2}\|\nabla\chi_k\|^2 + \varepsilon\sum_{m=1}^{k} h_m \|\Delta\chi_m\|^2 \le \frac{C_2}{2} - \sum_{m=1}^{k} h_m \int_\Omega L\nabla u_m \bullet \nabla\chi_m\,\mathrm{d}x\,, \tag{4.13}$$

where the constants C_2, C_3 are characterized in Lemma 4.1.

Proof. This proof uses ideas from Colli–Sprekels (see [CS94, Lemma 3.3] and [CS95, Lemma 3.1]). We define $\xi_{0\delta\varepsilon}$, χ_{-1}, h_0 as in Lemma 4.1. For $1 \le m \le K$ we can thus test the difference of (2.35) for m and $m-1$ by $\frac{\chi_m - \chi_{m-1}}{h_m}$. By applying Green's formula, (2.37), (2.36), the monotonicity of β, and Young's inequality, we obtain that

$$\begin{aligned} &\frac{\delta}{2}\left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|^2 + \varepsilon h_m \left\|\nabla\left(\frac{\chi_m - \chi_{m-1}}{h_m}\right)\right\|^2 \\ &\le \frac{\delta}{2}\left\|\frac{\chi_{m-1} - \chi_{m-2}}{h_{m-1}}\right\|^2 - \int_\Omega L(u_m - u_{m-1})\left(\frac{\chi_m - \chi_{m-1}}{h_m}\right)\mathrm{d}x\,. \end{aligned} \tag{4.14}$$

Summation from $m = 1$ to $m = k$ and use of (2.41) and (4.6) give the estimate (4.11).
To prove (4.12) and (4.13) rigorously, we use the Yosida approximation of β,

$$\beta_k(r) = \begin{cases} kr & , \text{ for } r < 0 \\ 0 & , \text{ for } 0 \le r \le 1 \\ k(r-1), & \text{ for } 1 < r \end{cases}, \quad \forall\, k \in \mathbb{N}. \tag{4.15}$$

Let $1 \le m \le K$ be arbitrary. Since $s \mapsto \int_0^s \beta_k(x)\,\mathrm{d}x$ is a continuous and convex function, it follows from [Bré71, Corollary 13] that for every $k \in \mathbb{N}$ there is a unique $\chi_{m,k}$ in $H^2(\Omega)$ such that

$$\delta\frac{\chi_{m,k} - \chi_{m-1}}{h_m} - \varepsilon\Delta\chi_{m,k} + \beta_k(\chi_{m,k}) = L(u_C - u_m) \quad \text{a.e. in} \quad \Omega, \tag{4.16}$$

$$\frac{\partial\chi_{m,k}}{\partial n} = 0 \quad \text{a.e. in} \quad \Gamma. \tag{4.17}$$

Since $\beta_k(\chi_{m,k})(\chi_{m,k}-\chi_{m-1}) \geq 0$, by (4.10) and (4.15), we obtain by testing (4.16) with $\chi_{m,k}-\chi_{m-1}$, applying Green's formula, (4.17), Hölder's, and Young's inequalities that

$$\delta h_m \left\| \frac{\chi_{m,k}-\chi_{m-1}}{h_m} \right\|^2 + \frac{\varepsilon}{2} \|\nabla \chi_{m,k}\|^2 \leq \frac{\varepsilon}{2} \|\nabla \chi_{m-1}\|^2 + \int_\Omega L(u_C - u_m)(\chi_{m,k}-\chi_{m-1})\, \mathrm{d}x\,. \quad (4.18)$$

Testing (4.16) by $-h_m \Delta \chi_{m,k}$, resp. $(\beta_k(\chi_{m,k}))^{p-1}$ for an even $p \in \mathbb{N}$, and using that $\beta_k'(\tau) \geq 0$ for all $\tau \in \mathbb{R}$, we obtain analogously that

$$\frac{\delta}{2} \|\nabla \chi_{m,k}\|^2 + \varepsilon h_m \|\Delta \chi_{m,k}\|^2 \;\leq\; \frac{\delta}{2} \|\nabla \chi_{m-1}\|^2 - h_m \int_\Omega L \nabla u_m \bullet \nabla \chi_{m,k}\, \mathrm{d}x\,, \quad (4.19)$$

$$\|\beta_k(\chi_{m,k})\|_p \;\leq\; \|L(u_C-u_m)\|_p \leq \sqrt[p]{|\Omega|}\, \|L(u_C-u_m)\|_{L^\infty(\Omega)}\,. \quad (4.20)$$

Thus, the sequence $(\chi_{m,k})_k$ is, by (4.17) and Lemma A.3, bounded in $H^2(\Omega)$, and the sequence $(\beta_k(\chi_{m,k}))_k$ is bounded in $L^\infty(\Omega)$, since (4.20) implies that, if p tends to infinity,

$$\|\beta_k(\chi_{m,k})\|_{L^\infty(\Omega)} \leq \|L(u_C-u_m)\|_{L^\infty(\Omega)}\,. \quad (4.21)$$

Therefore, we have, for some subsequences, $\chi_{m,k_n} \to \bar{\chi}_m$ weakly in $H^2(\Omega)$ and strongly in $L^2(\Omega)$, as well as $\beta_{k_n}(\chi_{m,k_n}) \to \bar{\xi}_m$ weakly–star in $L^\infty(\Omega)$. Thus,

$$\lim_{n,n'\to\infty} \int_\Omega \beta_{k_n}(\chi_{m,k_n})\, \chi_{m,k_{n'}}\, \mathrm{d}x \;=\; \int_\Omega \bar{\xi}_m \bar{\chi}_m\, \mathrm{d}x$$

and therefore, by [Bar76, Prob. 1.1(iv)], $\bar{\xi} \in \beta(\bar{\chi})$. Now, a passage to the limit in (4.16) and (4.17) yields that $(\bar{\chi}, \bar{\xi})$ is a solution to (2.35)–(2.37). Since this solution is unique by [Bré71, Corollary 13], we have $\chi_m = \bar{\chi}$ and $\xi_m = \bar{\xi}_m$. Thus, thanks to (4.19), (4.20), (4.21), and the weak–star lower semicontinuity of norms, we have that

$$\frac{\delta}{2} \|\nabla \chi_m\|^2 + \varepsilon h_m \|\Delta \chi_m\|^2 \;\leq\; \frac{\delta}{2} \|\nabla \chi_{m-1}\|^2 - h_m \int_\Omega L \nabla u_m \bullet \nabla \chi_m\, \mathrm{d}x \quad (4.22)$$

and (4.12) hold. Summation of (4.22) from $m=1$ to $m=k$ and use of (2.41), (4.4), and (4.1) give the estimate (4.13). □

Lemma 4.3 *There are three constants C_5, C_6, C_7, such that for $1 \leq k \leq K$,*

$$\begin{aligned}
&\frac{c_0}{2} \sum_{m=1}^k h_m \left\| \frac{u_m - u_{m-1}}{h_m \sqrt{u_m u_{m-1}}} \right\|^2 + C_5 \|u_k\|^2_{H^1(\Omega)} \\
\leq\;& C_6 \left(1 + \sum_{m=1}^{k-1} h_m \|u_m\|^2_{H^1(\Omega)} \right) + \sum_{m=1}^k h_m \int_\Omega L \frac{\chi_m - \chi_{m-1}}{h_m} \frac{u_m - u_{m-1}}{h_m}\, \mathrm{d}x\,, \quad (4.23) \\
&c_0 \sum_{m=1}^k h_m \left\| \frac{\nabla u_m}{u_m} \right\|^2 + \frac{\kappa}{4} \left\| \sum_{m=1}^k h_m \Delta u_m \right\|^2 \\
\leq\;& C_7 \left(1 + \sum_{m=1}^{k-1} h_m \left\| \sum_{i=1}^m h_i \Delta u_i \right\|^2 + \sum_{m=1}^k h_m \|u_m\|^2_\Gamma \right) + \sum_{m=1}^k h_m \int_\Omega L \nabla \chi_m \bullet \nabla u_m\, \mathrm{d}x\,. \quad (4.24)
\end{aligned}$$

Proof. Inserting $v = -(u_m - u_{m-1})$ in (2.42), applying (4.10), Young's inequality and (3.3), we get, after summation from $m = 1$ to $m = k$,

$$
\begin{aligned}
& c_o \sum_{m=1}^{k} h_m \left\| \frac{u_m - u_{m-1}}{h_m \sqrt{u_m u_{m-1}}} \right\|^2 + \frac{\kappa}{2} \|\nabla u_k\|^2 + \frac{c_\gamma}{2} \|u_k\|_\Gamma^2 \\
\leq \;& \frac{\kappa}{2} \|\nabla u_0\|^2 + \frac{1}{2} \sum_{m=1}^{k-1} h_m \int\limits_\Gamma u_m^2 \frac{\gamma_{m+1} - \gamma_m}{h_m} \,\mathrm{d}\sigma + \frac{1}{2} \int\limits_\Gamma u_0^2 \gamma_1 \,\mathrm{d}\sigma \\
& + \int\limits_\Gamma u_k \zeta_k \,\mathrm{d}\sigma + \sum_{m=1}^{k-1} h_m \int\limits_\Gamma u_m \frac{\zeta_m - \zeta_{m+1}}{h_m} \,\mathrm{d}\sigma - \int\limits_\Gamma u_0 \zeta_1 \,\mathrm{d}\sigma \\
& + L \sum_{m=1}^{k} h_m \int\limits_\Omega \frac{\chi_m - \chi_{m-1}}{h_m} \frac{u_m - u_{m-1}}{h_m} \,\mathrm{d}x - \sum_{m=1}^{k} h_m \int\limits_\Omega g_m \frac{u_m - u_{m-1}}{h_m} \,\mathrm{d}x\,. \qquad (4.25)
\end{aligned}
$$

Using (3.2) as well as the inequalities of Hölder and Young, yields, for every $\alpha^* > 0$,

$$
\begin{aligned}
& -\sum_{m=1}^{k} h_m \int\limits_\Omega g_m \frac{u_m - u_{m-1}}{h_m} \,\mathrm{d}x \leq C_a \sum_{m=1}^{k} h_m \left\| \frac{u_m - u_{m-1}}{h_m \sqrt{u_m u_{m-1}}} \right\| \sqrt{\|u_m\| \, \|u_{m-1}\|} \\
\leq \;& \frac{c_0}{2} \sum_{m=1}^{k} h_m \left\| \frac{u_m - u_{m-1}}{h_m \sqrt{u_m u_{m-1}}} \right\|^2 + \frac{1}{2c_0} C_a^2 \sum_{m=1}^{k-1} h_m \left(\frac{1}{2} \|u_m\|^2 + \frac{1}{2} \|u_{m-1}\|^2 \right) \\
& + \frac{1}{2c_0} C_a^2 h_k \left(\frac{\alpha^*}{2} \|u_k\|^2 + \frac{1}{2\alpha^*} \|u_{k-1}\|^2 \right). \qquad (4.26)
\end{aligned}
$$

Thus, (4.23) follows using (4.25), Lemma A.2, (3.5), (3.2), Young's inequality, (2.41), (4.3), (2.30) and (4.26), for $\alpha^* > 0$ chosen sufficiently small.
Next, we multiply (2.38) by h_m. Summing the resulting equation from $m = 1$ to $m = i$, applying (2.41) and (2.33), we find

$$
c_0 \theta_i + L \chi_i + \kappa \sum_{m=1}^{i} h_m \Delta u_m = e_{0\delta\varepsilon} + \sum_{m=1}^{i} h_m g_m. \qquad (4.27)
$$

We test this by $h_i \cdot \Delta u_i$, apply (2.39), Green's formula, (2.40), and take the sum from $i = 1$ to $i = k$, to obtain

$$
\begin{aligned}
& c_0 \sum_{i=1}^{k} h_i \int\limits_\Omega \left(\frac{\nabla u_i}{u_i} \right)^2 \,\mathrm{d}x + \kappa \int\limits_\Omega \left(\sum_{i=1}^{k} h_i \Delta u_i \sum_{m=1}^{i} h_m \Delta u_m \right) \,\mathrm{d}x + \frac{c_0}{\kappa} \sum_{i=1}^{k} h_i \int\limits_\Gamma \theta_i \zeta_i \,\mathrm{d}\sigma \\
= \;& \frac{1}{\kappa} \sum_{i=1}^{k} h_i \int\limits_\Gamma \left(c_0 \gamma_i + L \chi_i \left(\gamma_i u_i - \zeta_i \right) \right) \,\mathrm{d}\sigma + \sum_{i=1}^{k} h_i \int\limits_\Omega L \nabla \chi_i \bullet \nabla u_i \,\mathrm{d}x \\
& + \int\limits_\Omega \left(e_{0\delta\varepsilon} \sum_{i=1}^{k} h_i \Delta u_i \right) \,\mathrm{d}x + \int\limits_\Omega \left(\sum_{i=1}^{k} h_i \Delta u_i \sum_{m=1}^{i} h_m g_m \right) \,\mathrm{d}x\,. \qquad (4.28)
\end{aligned}
$$

Using Lemma A.4, Lemma 3.1, and (4.10), we get

$$
\begin{aligned}
&c_0 \sum_{i=1}^{k} h_i \left\| \frac{\nabla u_i}{u_i} \right\|^2 + \frac{\kappa}{2} \left\| \sum_{i=1}^{k} h_i \Delta u_i \right\|^2 + \frac{c_0 c_\zeta}{\kappa} \sum_{i=1}^{k} h_i \left\| \theta_i \right\|_{L^1(\Gamma)} \\
&\leq \ \frac{1}{\kappa} \sum_{i=1}^{k} h_i \int_\Gamma \left(c_0 C_a + L C_a u_i \right) \mathrm{d}\sigma + \sum_{i=1}^{k} h_i \int_\Omega L \nabla \chi_i \bullet \nabla u_i \, \mathrm{d}x \\
&\quad + \int_\Omega \left(\left(e_{0\delta\varepsilon} + \sum_{i=1}^{k} h_i g_i \right) \sum_{i=1}^{k} h_i \Delta u_i \right) \mathrm{d}x - \sum_{i=1}^{k-1} h_{i+1} \int_\Omega g_{i+1} \sum_{m=1}^{i} h_m \Delta u_m \, \mathrm{d}x \, .
\end{aligned}
$$

Now, (4.24) follows by applying Lemma 3.1, Young's inequality, (4.4), and (2.30).

□

Lemma 4.4 *There exists a constant C_8 such that*

$$
\begin{aligned}
&\varepsilon \sum_{m=1}^{K} h_m \left\| \nabla \left(\frac{\chi_m - \chi_{m-1}}{h_m} \right) \right\|^2 + \sum_{m=1}^{K} h_m \left\| \frac{u_m - u_{m-1}}{h_m \sqrt{u_m u_{m-1}}} \right\|^2 \\
&+ \max_{1 \leq m \leq K} \left(\delta \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|^2 + \left\| c_0 \frac{\theta_m - \theta_{m-1}}{h_m} + L \frac{\chi_m - \chi_{m-1}}{h_m} \right\|^2_{H^1(\Omega)^*} \right) \\
&+ \max_{0 \leq k \leq K} \left(\left\| \chi_k \right\|_{L^\infty(\Omega)} + \varepsilon \left\| \chi_k \right\|^2_{H^1(\Omega)} + \left\| u_k \right\|^2_{H^1(\Omega)} \right) \leq C_8.
\end{aligned} \tag{4.29}
$$

Proof. Applying the discrete version of Gronwall's Lemma to the sum of (4.11) and (4.23), we obtain (4.29) after recalling (2.42), (3.2), (4.10), (2.41), and Lemma 4.1. □

Lemma 4.5 *There is a constant C_9 such that*

$$
\max_{1 \leq m \leq K} \left\| \xi_m \right\| + \max_{0 \leq m \leq K} \left\| \varepsilon \chi_m \right\|_{H^2(\Omega)} \leq C_9. \tag{4.30}
$$

Proof. Using (4.12), we obtain from (4.29) a uniform bound for $\|\xi_m\|$. Comparing the terms in (2.35), we have, owing to (4.29) and (4.1), a uniform bound for $\|\varepsilon \Delta \chi_m\|$. Therefore, owing to the boundary condition (2.37) and the uniform bound for $\|\varepsilon \chi_m\|_{H^1(\Omega)}$ in (4.29), we can control $\|\varepsilon \chi_m\|_{H^2(\Omega)}$, using Lemma A.3. From (2.41), (4.4), and (4.1), we get a uniform bound for $\|\varepsilon \chi_0\|_{H^2(\Omega)}$. □

Lemma 4.6 *There exists a constant C_{10} such that*

$$
\begin{aligned}
&\max_{0 \leq k \leq K} \left(\delta \left\| \chi_k \right\|^2_{H^1(\Omega)} + \left\| \theta_k \right\|^2 \right) + \max_{1 \leq k \leq K} \left\| \sum_{m=1}^{k} h_m \Delta u_m \right\|^2 \\
&\quad + \varepsilon \sum_{m=0}^{K} h_m \left\| \chi_m \right\|^2_{H^2(\Omega)} + \sum_{m=0}^{K} h_m \left\| \frac{\nabla u_m}{u_m} \right\|^2 \leq C_{10}.
\end{aligned} \tag{4.31}
$$

Proof. Summation of the inequalities (4.13) and (4.24), use of (4.29), the discrete Gronwall inequality, Lemma 4.1, Lemma A.2, Lemma A.3, (2.37), (2.41), and (4.4) gives the estimate (4.31), since, by (4.27), (4.29), (4.4), and (3.2), the boundedness of $\left\| \sum_{m=1}^{k} h_m \Delta u_m \right\|^2$ implies the boundedness of $\|\theta_k\|^2$. □

Lemma 4.7 *There is a constant C_{11}, such that*

$$\sum_{m=1}^{K} h_m \left\| \frac{u_m - u_{m-1}}{h_m} \right\|^2_{L^{\frac{3}{2}}(\Omega)} + \sum_{m=1}^{K} h_m \left\| \frac{\theta_m - \theta_{m-1}}{h_m} \right\|^2_{L^1(\Omega)} \leq C_{11}. \tag{4.32}$$

Proof. Using Hölder's inequality, we obtain

$$\sum_{m=1}^{K} h_m \left\| \frac{u_m - u_{m-1}}{h_m} \right\|^2_{L^{\frac{3}{2}}(\Omega)} \leq \max_{0\leq k\leq K} \|u_k\|^2_{L^6(\Omega)} \sum_{m=1}^{K} h_m \left\| \frac{u_m - u_{m-1}}{h_m\sqrt{u_m u_{m-1}}} \right\|^2. \tag{4.33}$$

We have, by (4.10) and Schwarz's inequality,

$$\sum_{m=1}^{K} h_m \left\| \frac{\theta_m - \theta_{m-1}}{h_m} \right\|^2_{L^1(\Omega)} \leq \max_{0\leq k\leq K} \|\theta_k\|^2 \sum_{m=1}^{K} h_m \left\| \frac{u_m - u_{m-1}}{h_m\sqrt{u_m u_{m-1}}} \right\|^2. \tag{4.34}$$

Thus, (4.32) follows by (4.33), (4.34), (A.1), (4.29), and (4.31). □

Lemma 4.8 *There is a constant C_{12} such that*

$$\varepsilon \sum_{m=1}^{K} h_m \left\| \frac{\chi_m - \chi_{m-1}}{h_m} \right\|^2_{H^1(\Omega)} \leq C_{12}. \tag{4.35}$$

Proof. Picking $v = 1$ in (2.42) and squaring the result, we arrive by Hölder's and Schwarz's inequality at

$$\begin{aligned} L^2 \left| \int_\Omega \frac{\chi_m - \chi_{m-1}}{h_m}\,\mathrm{d}x \right|^2 \quad &\leq \quad 4c_0^2 \left\| \frac{\theta_m - \theta_{m-1}}{h_m} \right\|^2_{L^1(\Omega)} + 4\|\gamma_m\|^2_{L^2(\Gamma)} \|u_m\|^2_{L^2(\Gamma)} \\ &\quad + 4\|\zeta_m\|^2_{L^1(\Gamma)} + 4\|g_m\|^2 |\Omega|. \end{aligned}$$

Multiplying by h_m, summing the result from $m = 1$ to $m = k$, and applying (4.32), (3.2), (4.29), and Poincaré's inequality, (see [Zei90b, 53a]), yields (4.35). □

In the sequel, $L^\infty(\Omega)$–bounds for u_m and θ_m are derived, which are needed to improve the error estimates and the convergence results.

Lemma 4.9 *There is a constant C_{13} such that*

$$\left(\delta^{\frac{5}{4}} + \varepsilon^{\frac{1}{2}}\right)\left(\max_{0\leq k\leq K} \|u_k\|_{L^\infty(\Omega)} + \max_{1\leq k\leq K} \|\xi_k\|_{L^\infty(\Omega)}\right) + \left(\delta^{\frac{5}{2}} + \varepsilon\right) \sum_{m=1}^{K} h_m \left\| \frac{u_m - u_{m-1}}{h_m} \right\|^2 \leq C_{13}. \tag{4.36}$$

Proof. The $L^\infty(\Omega)$–bound is derived using Moser's technique as in [Lau93, Lau94, HSZ93, SZ93, HS]. From (4.10), we obtain, using an idea from [HS, Lemma 2.4],

$$\frac{\theta_m - \theta_{m-1}}{h_m} = -\frac{u_m - u_{m-1}}{h_m u_m^2} - \frac{u_m - u_{m-1}}{h_m u_m}\left(\frac{1}{u_{m-1}} - \frac{1}{u_m}\right) \leq -\frac{u_m - u_{m-1}}{h_m u_m^2}. \tag{4.37}$$

For every $p \in \mathbb{N}$, we have u_m^{p+1} in $H^1(\Omega)$, since $u_m \in H^2(\Omega)$ by (2.34). Taking $v = -u_m^{p+1}$ in (2.42), and applying (4.37), Lemma 3.1, and the inequalities of Schwarz and Young, we realize that

$$\begin{aligned}
&\frac{c_0}{p h_m}\left(\|u_m\|_p^p - \|u_{m-1}\|_p^p\right) + \kappa(p+1)\left\|u_m^{\frac{p}{2}}\nabla u_m\right\|^2 + \int_\Gamma \left(c_\gamma u_m^{p+2} - \frac{p+1}{p+2}c_\gamma u_m^{p+2}\right)\,\mathrm{d}\sigma \\
\leq\;& \int_\Omega f_m u_m^{p+1}\,\mathrm{d}x + \frac{1}{p+2}c_\gamma^{-(p+1)}\|\zeta_m\|_{L^\infty(\Gamma)}^{p+2}\int_\Gamma 1\,\mathrm{d}\sigma
\end{aligned} \tag{4.38}$$

with

$$f_m := L\frac{\chi_m - \chi_{m-1}}{h_m} - g_m. \tag{4.39}$$

Multiplying by $p h_m$ and taking the sum from $m=1$ to $m=k$, we derive from (3.2), Hölder's inequality, (2.41), and (4.3) that

$$\begin{aligned}
&c_0\|u_k\|_p^p + \kappa\frac{4p(p+1)}{(p+2)^2}\sum_{m=1}^k h_m\left\|\nabla u_m^{\frac{p+2}{2}}\right\|^2 + c_\gamma\frac{p}{p+2}\sum_{m=1}^k h_m\left\|u_m^{\frac{p+2}{2}}\right\|_\Gamma^2 \\
\leq\;& c_0\left(b+\frac{a}{2}\right)^p|\Omega| + p\sum_{m=1}^k h_m\int_\Omega f_m u_m^{p+1}\,\mathrm{d}x + T\frac{p}{p+2}c_\gamma^{-(p+1)}C_a^{p+2}\int_\Gamma 1\,\mathrm{d}\sigma\,.
\end{aligned} \tag{4.40}$$

Since the p–fractions on the left–hand side can be uniformly bounded from below by a positive constant, we have, by Lemma A.2,

$$c_0\|u_k\|_p^p + C_{14}\sum_{m=1}^k h_m\left\|u_m^{\frac{p+2}{2}}\right\|_{H^1(\Omega)}^2 \leq C_{15}^{p+2} + p\sum_{m=1}^k h_m\int_\Omega f_m u_m^{p+1}\,\mathrm{d}x\,. \tag{4.41}$$

First, we will estimate this analogously to Laurençot [Lau94, Lemma 2.3], using the fact that, by (4.39), (4.35), (A.1), (3.2), and (4.1),

$$\varepsilon\sum_{m=1}^K h_m\|f_m\|_6^2 \leq C_{16}. \tag{4.42}$$

Using Hölder's inequality, (A.1), and Young's inequality, we obtain

$$\begin{aligned}
p\int_\Omega f_m u_m^{p+1}\,\mathrm{d}x \;&\leq\; p\|f_m\|_6\left(\int_\Omega u_m^{\frac{3p}{5}}u_m^{\frac{3}{5}(p+2)}\,\mathrm{d}x\right)^{\frac{5}{6}} \leq p\|f_m\|_6\left\|u_m^{\frac{3p}{5}}\right\|_{\frac{5}{4}}^{\frac{5}{6}}\left\|u_m^{\frac{p+2}{2}}\right\|_6 \\
&\leq\; C_{17}p^2\|f_m\|_6^2\|u_m\|_{\frac{3}{4}p}^p + \frac{C_{14}}{2}\left\|u_m^{\frac{p+2}{2}}\right\|_{H^1(\Omega)}^2.
\end{aligned}$$

Therefore we have, by (4.41) and (4.42),

$$c_0\|u_k\|_p^p + \frac{C_{14}}{2}\sum_{m=1}^k h_m\left\|u_m^{\frac{p+2}{2}}\right\|_{H^1(\Omega)}^2 \leq C_{15}^p C_{15}^2 + C_{17}p^2\frac{C_{16}}{\varepsilon}\max_{1\leq m\leq k}\|u_m\|_{\frac{3}{4}p}^p\,.$$

For $p_0 = 6$, $p_{n+1} = \frac{4}{3}p_n$ now holds, by (4.1),

$$\max_{1\le k\le K} \|u_k\|_{p_n}^{p_n} \le C_{18}\frac{1}{\varepsilon}p_n^2 \max\left\{C_{19}^{p_n}, \left(\max_{1\le k\le K}\|u_k\|_{p_{n-1}}^{p_{n-1}}\right)^{\frac{4}{3}}\right\},$$

and, by (4.29) and (A.1),

$$\max_{1\le k\le K} \|u_k\|_{p_0}^{p_0} \le C_{20}. \tag{4.43}$$

Thus, we have, by Lemma A.5,

$$\varepsilon^{\frac{1}{2}} \max_{1\le k\le K} \|u_k\|_{L^\infty(\Omega)} \le C_{21}. \tag{4.44}$$

Next, we will use an estimation similar to Horn–Sprekels–Zheng [HSZ93]. We define $p_0 = 6$, $p_{n+1} = 2p_n - 2$. Recalling (4.41) with $p = p_n$, Schwarz's inequality, (4.39), (4.29), (3.2), and (4.1), we see that

$$c_0 \|u_k\|_{p_n}^{p_n} + C_{14}\sum_{m=1}^{k} h_m \left\|u_m^{p_{n-1}}\right\|_{H^1(\Omega)}^2 \le C_{15}^{p_n+2} + p_n C_{22}\frac{1}{\sqrt{\delta}}\sum_{m=1}^{k} h_m \left\|u_m^{p_n+1}\right\|. \tag{4.45}$$

Setting $q_n := \frac{p_n+1}{p_{n-1}}$, we have

$$q_n = 2 - \frac{1}{p_{n-1}} \in \left[\frac{11}{6}, 2\right), \tag{4.46}$$

and applying Hölder's and Nirenberg–Gagliardo's (see [Zei90b, Chapter 54 a]) inequalities, we can conclude that

$$\begin{aligned}
\left\|u_m^{p_n+1}\right\| &= \left(\int_\Omega u_m^{2p_{n-1}q_n}\,\mathrm{d}x\right)^{\frac{1}{2}} \le \left(|\Omega|^{\frac{2-q_n}{2}}\left(\int_\Omega u_m^{4p_{n-1}}\,\mathrm{d}x\right)^{\frac{q_n}{2}}\right)^{\frac{1}{2}} \\
&\le |\Omega|^{\frac{2-q_n}{4}} \left\|u_m^{p_{n-1}}\right\|_4^{q_n} \le C_{23}\left\|u_m^{p_{n-1}}\right\|_{H^1(\Omega)}^{\frac{9q_n}{10}} \left\|u_m^{p_{n-1}}\right\|_{L^1(\Omega)}^{\frac{q_n}{10}}.
\end{aligned} \tag{4.47}$$

Hence, by Young's inequality,

$$p_n C_{22}\frac{1}{\sqrt{\delta}}\left\|u_m^{p_n+1}\right\| \le \frac{9q_n}{20}\varepsilon_*^{\frac{20}{9q_n}}\left\|u_m^{p_{n-1}}\right\|_{H^1(\Omega)}^2 + \frac{20-9q_n}{20}\left(p_n C_{22}C_{23}\frac{1}{\varepsilon_*\sqrt{\delta}}\right)^{\frac{20}{20-9q_n}}\left\|u_m^{p_{n-1}}\right\|_{L^1(\Omega)}^{\frac{2q_n}{20-9q_n}}, \tag{4.48}$$

where we choose $\varepsilon_* > 0$ such that

$$\frac{9q_n}{20}\varepsilon_*^{\frac{20}{9q_n}} = \frac{C_{14}}{2}, \qquad \text{i.e.} \quad \varepsilon_* = \left(\frac{10C_{14}}{9q_n}\right)^{\frac{9q_n}{20}}.$$

Recalling (4.46) and (4.1), we find that

$$\frac{q_n}{20-9q_n} \le \frac{2}{20-9q_n} \le 1,$$

$$\left(p_n C_{22}C_{23}\frac{1}{\sqrt{\delta}}\right)^{\frac{20}{20-9q_n}} \le p_n^{10}C_{24}\delta^{-5} \quad , \quad \varepsilon_*^{\frac{-20}{20-9q_n}} \le \left(\frac{18}{C_{14}10}\right)^{\frac{9q_n}{20-9q_n}} \le C_{25}.$$

Therefore, by (4.48),

$$p_n C_{22} \frac{1}{\sqrt{\delta}} \left\| u_m^{p_n+1} \right\| \leq \frac{C_{14}}{2} \left\| u_m^{p_{n-1}} \right\|_{H^1(\Omega)}^2 + p_n^{10} C_{24} \delta^{-5} C_{25} \max\left\{1, \left\| u_m^{p_{n-1}} \right\|_{L^1(\Omega)}^2\right\}. \tag{4.49}$$

Thus, recalling (4.45), we have

$$\max_{1\leq k\leq K} \|u_k\|_{p_n}^{p_n} \leq \frac{1}{c_0} C_{15}^{p_n+2} + \frac{1}{c_0} p_n^{10} \delta^{-5} C_{24} C_{25} T \max_{1\leq k\leq K} \max\left\{1, \left(\|u_k\|_{p_{n-1}}^{p_{n-1}}\right)^2\right\}. \tag{4.50}$$

Recalling (4.43), we can now use Lemma A.5 to obtain

$$\delta^{\frac{5}{4}} \max_{1\leq k\leq K} \|u_k\|_{L^\infty(\Omega)} \leq C_{26}. \tag{4.51}$$

Since we have the $L^\infty(\Omega)$–bounds for u_m in (4.44) and (4.51), to verify (4.36) it suffices to use (4.12) and a calculation analogous to the first one in the proof of Lemma 4.7. □

Lemma 4.10 *There is a positive constant C_{27} such that*

$$\begin{aligned} &\varepsilon^3 \max_{0\leq k\leq K} \left(\|\theta_k\|_{L^\infty(\Omega)}^2 + \|\theta_k\|_{H^1(\Omega)}\right) \\ &+ \varepsilon^3 \sum_{m=1}^K h_m \left(\left\|\frac{\theta_m - \theta_{m-1}}{h_m}\right\|^2 + \|u_m\|_{H^2(\Omega)}^2\right) + \varepsilon^3 \sum_{m=0}^K \|\theta_m\|_{H^1(\Omega)}^2 \leq C_{27}. \end{aligned} \tag{4.52}$$

Proof.
Since the calculation for the $L^\infty(\Omega)$–bounds are quite similar to the estimates analogous to Laurençot [Lau94] used in the last lemma, we will only give a sketch of the proof.
After inserting $v = \theta_m^p$ in (2.42), we get for $p = p_n + 1$, with $p_0 = 2$, $p_{n+1} = \frac{4}{3}p_n$,

$$\max_{1\leq k\leq K} \|\theta_k\|_{p_n}^{p_n} \leq \frac{C_{28}}{\varepsilon} p_n^2 \max\left\{C_{29}^{p_n}, \left(\max_{1\leq k\leq K} \|\theta_k\|_{p_{n-1}}^{p_{n-1}}\right)^{\frac{4}{3}}\right\}.$$

Now we have by (4.31), Lemma A.5, (2.41), (4.3), and (4.1) that

$$\varepsilon^{\frac{3}{2}} \max_{0\leq k\leq K} \|\theta_k\|_{L^\infty(\Omega)} \leq C_{30}. \tag{4.53}$$

Recalling (4.10) and (4.29), we see by Hölder's inequality that

$$\varepsilon^3 \sum_{m=1}^K h_m \left\|\frac{\theta_m - \theta_{m-1}}{h_m}\right\|^2 \leq \varepsilon^3 \max_{0\leq m\leq K} \|\theta_m\|_{L^\infty(\Omega)}^2 \sum_{m=1}^K h_m \left\|\frac{u_m - u_{m-1}}{h_m\sqrt{u_m u_{m-1}}}\right\|^2 \leq C_{30}^2 C_{11}. \tag{4.54}$$

Therefore, thanks to (2.38), (4.29), (4.1), and (3.2), we have $\varepsilon^3 \sum_{m=1}^K h_m \|\Delta u_m\|^2 \leq C_{31}$. Using Lemma A.3, (4.29), (3.4), and (3.2), we obtain

$$\varepsilon^3 \sum_{m=1}^K h_m \|u_m\|_{H^2(\Omega)}^2 \leq C_{32}. \tag{4.55}$$

Moreover, we have by (4.53), (4.29), and (4.31)

$$\varepsilon^3 \max_{0\leq m\leq K} \|\nabla\theta_m\| + \varepsilon^3 \sum_{m=1}^k h_m \|\nabla\theta_m\|^2 \leq C_{30}^2 C_8 + C_{30}^2 C_{10}. \tag{4.56}$$

Thus, (4.52) follows from (4.53)–(4.56), (4.31), and (4.1).

□

5 Error estimates and convergence

In this chapter, we will derive error estimates for our scheme and prove the convergence of the approximations. We will use ideas from an error estimate for the relaxed–in–time Stefan problem by Colli in [Col95].
In the sequel, we suppose that the assumptions corresponding to one of the Theorems 2.3 to 2.6 are satisfied, and define (θ, u, χ, ξ) as in the respective theorem.
We consider $\delta > 0$, $\varepsilon > 0$ with (4.1), and admissible decompositions Z. By Lemma 4.1 and Theorem 2.2, there exists a unique solution to $(\mathbf{D}_{Z,\delta,\varepsilon})$ that defines a corresponding approximation $\left(\widehat{\theta}^{Z\delta\varepsilon}, \widehat{u}^{Z\delta\varepsilon}, \widehat{\chi}^{Z\delta\varepsilon}, \overline{\xi}^{Z\delta\varepsilon}\right)$, as in Remark 2.2. We define $\overline{\theta}^{Z\delta\varepsilon}$, $\overline{u}^{Z\delta\varepsilon}$, and $\overline{\chi}^{Z\delta\varepsilon}$ analogously to $\overline{\xi}^{Z\delta\varepsilon}$.
For the functions g_m, γ_m, ζ_m, defined in (2.31) and (2.32), we define $g^Z \in L^\infty(\Omega_T)$, γ^Z, $\zeta^Z \in L^\infty(\Sigma)$, analogously to $\overline{\xi}^{Z\delta\varepsilon}$. Then, we have by the definition of the approximations, (2.34)–(2.37), (2.39), (2.42), and (2.57) (see Lemma 4.1),

$$\widehat{\chi}^{Z\delta\varepsilon} \in H^1(0,T;L^2(\Omega)),\ \widehat{\chi}^{Z\delta\varepsilon}, \overline{\chi}^{Z\delta\varepsilon} \in L^\infty(0,T;H^2(\Omega)),\ \overline{\xi}^{Z\delta\varepsilon} \in L^\infty(0,T;L^2(\Omega)), \tag{5.1}$$

$$\widehat{\theta}^{Z\delta\varepsilon}, \widehat{u}^{Z\delta\varepsilon} \in H^1(0,T;L^2(\Omega)),\ \overline{u}^{Z\delta\varepsilon} \in L^2(0,T;H^2(\Omega)),\ \widehat{u}^{Z\delta\varepsilon} \in L^2(|Z|,T;H^2(\Omega)), \tag{5.2}$$

$$\delta\widehat{\chi}_t^{Z\delta\varepsilon} - \varepsilon\Delta\overline{\chi}^{Z\delta\varepsilon} + \overline{\xi}^{Z\delta\varepsilon} = L\left(u_C - \overline{u}^{Z\delta\varepsilon}\right) \quad \text{a.e. in} \quad \Omega_T, \tag{5.3}$$

$$\begin{aligned} &\int_\Omega \left(c_0\widehat{\theta}_t^{Z\delta\varepsilon}(t) + L\widehat{\chi}_t^{Z\delta\varepsilon}(t)\right) v \,\mathrm{d}x - \kappa \int_\Omega \nabla\overline{u}^{Z\delta\varepsilon}(t) \bullet \nabla v \,\mathrm{d}x \\ &- \int_\Gamma \left(\gamma^Z(t)\overline{u}^{Z\delta\varepsilon}(t) - \zeta^Z(t)\right) v \,\mathrm{d}\sigma = \int_\Omega g^Z(t) v \,\mathrm{d}x\,, \quad \forall\, v \in H^1(\Omega), \quad \text{for a.e. } t \in (0,T), \end{aligned} \tag{5.4}$$

$$\overline{\xi}^{Z\delta\varepsilon} \in \beta\left(\overline{\chi}^{Z\delta\varepsilon}\right), \quad 0 < \overline{u}^{Z\delta\varepsilon}, \quad \overline{\theta}^{Z\delta\varepsilon} = \frac{1}{\overline{u}^{Z\delta\varepsilon}} \quad \text{a.e. in} \quad \Omega_T, \tag{5.5}$$

$$\frac{\partial\widehat{\chi}^{Z\delta\varepsilon}}{\partial n} = 0, \quad \frac{\partial\overline{\chi}^{Z\delta\varepsilon}}{\partial n} = 0 \quad \text{a.e. in} \quad \Sigma, \tag{5.6}$$

$$\widehat{\theta}^{Z\delta\varepsilon}(\cdot,0) = \theta_{0\delta\varepsilon}, \quad \widehat{u}^{Z\delta\varepsilon}(\cdot,0) = u_{0\delta\varepsilon}, \quad \widehat{\chi}^{Z\delta\varepsilon}(\cdot,0) = \chi_{0\delta\varepsilon} \qquad \text{a.e. in} \quad \Omega. \tag{5.7}$$

In the sequel, C_i^* will always denote positive generic constants, independent of the approximation, and the decomposition Z, as well as independent of δ (resp. ε), if δ (resp. ε) is not a fixed parameter.
We find from Lemma 4.4 to Lemma 4.10 that

$$\begin{aligned} &\left\|\overline{\chi}^{Z\delta\varepsilon}\right\|_{L^\infty(\Omega_T)} + \left\|\widehat{\chi}^{Z\delta\varepsilon}\right\|_{L^\infty(\Omega_T)} + \varepsilon\left(\left\|\overline{\chi}^{Z\delta\varepsilon}\right\|_{L^\infty(0,T;H^2(\Omega))} + \left\|\overline{\chi}^{Z\delta\varepsilon}\right\|^2_{L^2(0,T;H^2(\Omega))}\right) \\ &\quad + \varepsilon\left(\left\|\widehat{\chi}^{Z\delta\varepsilon}\right\|_{L^\infty(0,T;H^2(\Omega))} + \left\|\widehat{\chi}^{Z\delta\varepsilon}\right\|^2_{L^2(0,T;H^2(\Omega))} + \left\|\widehat{\chi}^{Z\delta\varepsilon}\right\|^2_{H^1(0,T;H^1(\Omega))}\right) \\ &\quad + \delta\left(\left\|\widehat{\chi}^{Z\delta\varepsilon}\right\|^2_{W^{1,\infty}(0,T;L^2(\Omega))} + \left\|\widehat{\chi}^{Z\delta\varepsilon}\right\|^2_{L^\infty(0,T;H^1(\Omega))} + \left\|\overline{\chi}^{Z\delta\varepsilon}\right\|^2_{L^\infty(0,T;H^1(\Omega))}\right) \le C_1^*, \end{aligned} \tag{5.8}$$

$$\left\|\overline{\xi}^{Z\delta\varepsilon}\right\|^2_{L^\infty(0,T;L^2(\Omega))} + \left(\delta^{\frac{5}{4}} + \varepsilon^{\frac{1}{2}}\right)\left\|\overline{\xi}^{Z\delta\varepsilon}\right\|_{L^\infty(\Omega_T)} \le C_2^*, \tag{5.9}$$

$$\begin{aligned} &\left\|\widehat{u}^{Z\delta\varepsilon}\right\|^2_{H^1(0,T;L^{\frac{3}{2}}(\Omega))} + \left\|\widehat{u}^{Z\delta\varepsilon}\right\|^2_{L^\infty(0,T;H^1(\Omega))} + \left\|\overline{u}^{Z\delta\varepsilon}\right\|^2_{L^\infty(0,T;H^1(\Omega))} \\ &\quad + \left(\delta^{\frac{5}{2}} + \varepsilon\right)\left(\left\|\widehat{u}^{Z\delta\varepsilon}\right\|^2_{H^1(0,T;L^2(\Omega))} + \left\|\widehat{u}^{Z\delta\varepsilon}\right\|^2_{L^\infty(\Omega_T)} + \left\|\overline{u}^{Z\delta\varepsilon}\right\|^2_{L^\infty(\Omega_T)}\right) \end{aligned}$$

$$+ \varepsilon^3 \left(\left\| \widehat{u}^{Z\delta\varepsilon} \right\|^2_{L^2(|Z|,T;H^2(\Omega))} + \left\| \overline{u}^{Z\delta\varepsilon} \right\|^2_{L^2(0,T;H^2(\Omega))} \right) \leq C_3^*, \tag{5.10}$$

$$\begin{aligned} &\left\| \widehat{\theta}^{Z\delta\varepsilon} \right\|^2_{H^1(0,T;L^1(\Omega))} + \left\| \widehat{\theta}^{Z\delta\varepsilon} \right\|^2_{L^\infty(0,T;L^2(\Omega))} + \left\| \overline{\theta}^{Z\delta\varepsilon} \right\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\quad + \varepsilon^3 \left(\left\| \widehat{\theta}^{Z\delta\varepsilon} \right\|^2_{H^1(0,T;L^2(\Omega))} + \left\| \widehat{\theta}^{Z\delta\varepsilon} \right\|^2_{L^\infty(\Omega_T)} + \left\| \overline{\theta}^{Z\delta\varepsilon} \right\|^2_{L^\infty(\Omega_T)} \right) \\ &\quad + \varepsilon^3 \left(\left\| \widehat{\theta}^{Z\delta\varepsilon} \right\|_{L^\infty(0,T;H^1(\Omega))} + \left\| \overline{\theta}^{Z\delta\varepsilon} \right\|_{L^\infty(0,T;H^1(\Omega))} \right) \leq C_4^*, \end{aligned} \tag{5.11}$$

$$\left\| c_0 \widehat{\theta}_t^{Z\delta\varepsilon} + L \widehat{\chi}_t^{Z\delta\varepsilon} \right\|^2_{L^\infty(0,T;H^1(\Omega)^*)} \leq C_5^*. \tag{5.12}$$

The difference between the piecewise linear and the piecewise constant approximations can be estimated using Lemmas 4.4 and 4.6 to 4.10,

$$\delta \left\| \widehat{\chi}^{Z\delta\varepsilon} - \overline{\chi}^{Z\delta\varepsilon} \right\|^2_{L^\infty(0,T;L^2(\Omega))} + \varepsilon \left\| \widehat{\chi}^{Z\delta\varepsilon} - \overline{\chi}^{Z\delta\varepsilon} \right\|^2_{L^2(0,T;H^1(\Omega))} \leq |Z|^2 C_6^*, \tag{5.13}$$

$$\left\| \widehat{u}^{Z\delta\varepsilon} - \overline{u}^{Z\delta\varepsilon} \right\|^2_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} + \left(\delta^{\frac{5}{2}} + \varepsilon \right) \left\| \widehat{u}^{Z\delta\varepsilon} - \overline{u}^{Z\delta\varepsilon} \right\|^2_{L^2(0,T;L^2(\Omega))} \leq |Z|^2 C_7^*, \tag{5.14}$$

$$\left\| \widehat{\theta}^{Z\delta\varepsilon} - \overline{\theta}^{Z\delta\varepsilon} \right\|^2_{L^2(0,T;L^1(\Omega))} + \varepsilon^3 \left\| \widehat{\theta}^{Z\delta\varepsilon} - \overline{\theta}^{Z\delta\varepsilon} \right\|^2_{L^2(0,T;L^2(\Omega))} \leq |Z|^2 C_8^*, \tag{5.15}$$

$$\left\| c_0 \widehat{\theta}^{Z\delta\varepsilon} + L \widehat{\chi}^{Z\delta\varepsilon} - \left(c_0 \overline{\theta}^{Z\delta\varepsilon} + L \overline{\chi}^{Z\delta\varepsilon} \right) \right\|^2_{L^\infty(0,T;H^1(\Omega)^*)} \leq |Z|^2 C_9^*. \tag{5.16}$$

For the data, we have the following estimates.

Lemma 5.1

$$\left\| g - g^Z \right\|_{L^2(0,T;L^\infty(\Omega))} + \left\| \gamma - \gamma^Z \right\|_{L^\infty(\Sigma)} + \left\| \zeta - \zeta^Z \right\|_{L^\infty(\Sigma)} \leq C_{10}^* |Z|, \tag{5.17}$$

$$\gamma^Z \geq c_\gamma, \qquad \zeta^Z \geq c_\zeta \quad a.e. \ in \quad \Sigma. \tag{5.18}$$

Proof. From (2.22)–(2.25), (2.31), and (2.32), one can derive elementary (5.17). The lower bound (5.18) follows immediately from (3.3). □

Remark 5.1 If in (2.25) the assumption $g_t \in L^2(0,T;L^\infty(\Omega))$ is omitted, then the estimate for g^Z in (5.17) is lost. But, using the p–mean value theorem (see [Zei90a, Prob. 23.9]), we still could prove $g^Z \to g$ strongly in $L^2(0,T;L^\infty(\Omega))$.

Now, we estimate the difference between the approximation and the exact solution. First, we work on the equation for θ and u. We obtain from (2.2), (2.3), (2.11) and (2.26), resp. (2.12) and (2.14),

$$\theta \in L^\infty(0,T;L^2(\Omega)), \quad u \in L^\infty(0,T;H^1(\Omega)), \quad (c_o\theta + L\chi)(0) = e_s \in H^1(\Omega)^*. \tag{5.19}$$

Integration in time of the difference between (5.4) and (2.8) yields, by (5.7) and (2.33), for a.e. $t \in (0,T)$ and for all $v \in H^1(\Omega)$,

$$\int_\Omega \left(c_0\widehat{\theta}^{Z\delta\varepsilon}(t) + L\widehat{\chi}^{Z\delta\varepsilon}(t) - (c_0\theta(t) + L\chi(t))\right) v \,\mathrm{d}x \;-\int_\Omega (e_{0\delta\varepsilon} - e_s)\, v\,\mathrm{d}x$$

$$= \;\kappa \int_0^t\int_\Omega \nabla\left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right) \bullet \nabla v\,\mathrm{d}x\;\mathrm{d}\tau \;+ \int_0^t\int_\Omega \left(g^Z(\tau) - g(\tau)\right) v\,\mathrm{d}x\;\mathrm{d}\tau$$

$$+ \int_0^t\int_\Gamma \left(\gamma^Z(\tau)\overline{u}^{Z\delta\varepsilon}(\tau) - \zeta^Z(\tau) - (\gamma(\tau)u(\tau) - \zeta(\tau))\right) v\,\mathrm{d}\sigma\;\mathrm{d}\tau\,. \tag{5.20}$$

With $v = -\left(\overline{u}^{Z\delta\varepsilon}(t) - u(t)\right)$, this yields

$$- \int_\Omega \left(c_0\widehat{\theta}^{Z\delta\varepsilon} + L\widehat{\chi}^{Z\delta\varepsilon} - (c_0\theta + L\chi)\right)\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}x \;+ \int_\Omega (e_{0\delta\varepsilon} - e_s)\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}x$$

$$= \; -\kappa\int_\Omega\int_0^t \nabla\left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau \bullet \nabla\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}x$$

$$-\int_\Gamma\int_0^t \left(\gamma^Z(\tau)\overline{u}^{Z\delta\varepsilon}(\tau) - \zeta^Z(\tau) - (\gamma(\tau)u(\tau) - \zeta(\tau))\right)\,\mathrm{d}\tau\;\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}\sigma$$

$$-\int_\Omega\int_0^t \left(g^Z(\tau) - g(\tau)\right)\,\mathrm{d}\tau\;\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}x \;=: -A_1 - A_2 - A_3. \tag{5.21}$$

We have

$$A_1 \;=\; \frac{\kappa}{2}\partial_t\left\|\nabla\int_0^t\left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\right\|^2. \tag{5.22}$$

Moreover, using (5.21), Schwarz's inequality, (5.17), the trace theorem for $H^1(\Omega)$–functions, (5.10), and (5.19), we obtain

$$\begin{aligned} A_2 \;&= \int_\Gamma\int_0^t\left(\gamma^Z(\tau) - \gamma(\tau)\right)\overline{u}^{Z\delta\varepsilon}(\tau)\,\mathrm{d}\tau\;\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}\sigma \\ &+ \int_\Gamma\int_0^t \gamma(\tau)\left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\;\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}\sigma \;- \int_\Gamma\int_0^t\left(\zeta^Z(\tau) - \zeta(\tau)\right)\,\mathrm{d}\tau\;\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}\sigma \\ &\geq \int_\Gamma \frac{1}{2\gamma}\partial_t\left(\int_0^t\gamma(\tau)\left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\right)^2\,\mathrm{d}\sigma \;- C_{11}^*\,|Z|\,. \end{aligned} \tag{5.23}$$

Finally, using (5.21), Hölder's inequality, (5.17), (5.10), (5.19), and Young's inequality, it follows that

$$\begin{aligned} -A_3 \;&\leq\; \left\|\int_0^t\left(g^Z(\tau) - g(\tau)\right)\,\mathrm{d}\tau\right\|_{L^\infty(\Omega)}\left\|\overline{u}^{Z\delta\varepsilon} - u\right\|_{L^1(\Omega)} \\ &\leq\; \sqrt{T}C_{10}^*\,|Z|\left\|\frac{\overline{u}^{Z\delta\varepsilon} - u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|\left\|\sqrt{\overline{u}^{Z\delta\varepsilon}u}\right\| \leq C_{12}^*\,|Z|^2 + \frac{c_0}{2}\left\|\frac{\overline{u}^{Z\delta\varepsilon} - u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|^2. \end{aligned} \tag{5.24}$$

Recalling (2.6) and (5.5), we see that

$$
\begin{aligned}
&-\int\limits_\Omega \left(c_0\widehat{\theta}^{Z\delta\varepsilon} + L\widehat{\chi}^{Z\delta\varepsilon} - (c_0\theta + L\chi)\right)\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}x \\
&= \quad -A_4 + c_0\int\limits_\Omega \frac{\left(\overline{u}^{Z\delta\varepsilon} - u\right)^2}{\overline{u}^{Z\delta\varepsilon}u}\,\mathrm{d}x - L\int\limits_\Omega \left(\overline{\chi}^{Z\delta\varepsilon} - \chi\right)\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}x
\end{aligned}
\tag{5.25}
$$

with

$$
A_4 := \int\limits_\Omega \left(c_0\widehat{\theta}^{Z\delta\varepsilon} + L\widehat{\chi}^{Z\delta\varepsilon} - \left(c_0\overline{\theta}^{Z\delta\varepsilon} + L\overline{\chi}^{Z\delta\varepsilon}\right)\right)\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}x\,. \tag{5.26}
$$

Moreover, using (5.16), (5.10), and (5.19), we obtain

$$
A_4 \le \left\| c_0\widehat{\theta}^{Z\delta\varepsilon} + L\widehat{\chi}^{Z\delta\varepsilon} - \left(c_0\overline{\theta}^{Z\delta\varepsilon} + L\overline{\chi}^{Z\delta\varepsilon}\right)\right\|_{H^1(\Omega)^*}\left\|\overline{u}^{Z\delta\varepsilon} - u\right\|_{H^1(\Omega)} \le |Z|\,C_{13}^*. \tag{5.27}
$$

Hence, we obtain from (5.21), Schwarz's inequality, (5.22)–(5.24), (5.25), and (5.27) that

$$
\begin{aligned}
&\frac{c_0}{2}\left\|\frac{\overline{u}^{Z\delta\varepsilon} - u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|^2 + \frac{\kappa}{2}\partial_t\left\|\nabla\int\limits_0^t \left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\right\|^2 \\
&+\int\limits_\Gamma \frac{1}{2\gamma}\partial_t\left(\int\limits_0^t \gamma(\tau)\left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\right)^2\,\mathrm{d}\sigma \\
\le\;& L\int\limits_\Omega \left(\overline{\chi}^{Z\delta\varepsilon} - \chi\right)\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}x + \|e_{0\delta\varepsilon} - e_s\|\left\|\overline{u}^{Z\delta\varepsilon} - u\right\| + |Z|\left(C_{11}^* + C_{13}^*\right) + C_{12}^*\,|Z|^2\,.
\end{aligned}
\tag{5.28}
$$

Thus, integrating over t and by parts, we find from Hölder's inequality and (2.23) that

$$
\begin{aligned}
&\frac{c_0}{2}\int\limits_0^s\left\|\frac{\overline{u}^{Z\delta\varepsilon} - u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|^2\,\mathrm{d}t + \frac{\kappa}{2}\left\|\nabla\int\limits_0^s \left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\right\|^2 \\
&\quad+\int\limits_\Gamma \frac{1}{2\gamma}\left(\int\limits_0^s \gamma(\tau)\left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\right)^2\,\mathrm{d}\sigma \\
&\le\; L\int\limits_0^s\int\limits_\Omega \left(\overline{\chi}^{Z\delta\varepsilon} - \chi\right)\left(\overline{u}^{Z\delta\varepsilon} - u\right)\,\mathrm{d}x\,\mathrm{d}t + \|e_{0\delta\varepsilon} - e_s\|\sqrt{T}\left\|\overline{u}^{Z\delta\varepsilon} - u\right\|_{L^2(0,T;L^2(\Omega))} \\
&\quad+ |Z|\,C_{14}^* + \left\|\frac{\gamma_t}{\gamma}\right\|_{L^\infty(\Sigma)}\int\limits_0^s\int\limits_\Gamma \frac{1}{2\gamma}\left(\int\limits_0^t \gamma(\tau)\left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\right)^2\,\mathrm{d}\sigma\,\mathrm{d}t\,.
\end{aligned}
\tag{5.29}
$$

We define δ', ε' in the framework of Theorem 2.3 by $\varepsilon' = \varepsilon$ and $\delta' = \delta$, in Theorem 2.4 by $\varepsilon' = 0$ and $\delta' = \delta$, in Theorem 2.5 by $\varepsilon' = \varepsilon$ and $\delta' = 0$, and in Theorem 2.6 by $\varepsilon' = 0$ and $\delta' = 0$. Then, we have by (2.4), (2.15), or (2.18),

$$
\delta'\chi \in W^{1,\infty}(0,T;L^2(\Omega)), \qquad \varepsilon'\chi \in L^\infty(0,T;H^2(\Omega)), \qquad \chi \in L^\infty\left(\Omega_T\right). \tag{5.30}
$$

It follows from (4.1)

$$0 \le \delta' \le \delta \le \overline{\delta}, \qquad 0 \le \varepsilon' \le \varepsilon \le \overline{\varepsilon}. \tag{5.31}$$

Subtracting (2.9), (2.16), (2.19) or (2.21), from (5.3), we have that a.e. in Ω_T,

$$\delta'\left(\widehat{\chi}_t^{Z\delta\varepsilon} - \chi_t\right) - \varepsilon'\Delta\left(\overline{\chi}^{Z\delta\varepsilon} - \chi\right) + \overline{\xi}^{Z\delta\varepsilon} - \xi = -L\left(\overline{u}^{Z\delta\varepsilon} - u\right) + (\delta' - \delta)\widehat{\chi}_t^{Z\delta\varepsilon} - (\varepsilon' - \varepsilon)\Delta\overline{\chi}^{Z\delta\varepsilon}. \tag{5.32}$$

From (2.10), (2.20), (2.11), and (2.17), we find that always

$$\varepsilon'\frac{\partial\chi}{\partial n} = 0 \quad \text{a.e. in} \quad \Sigma, \qquad \delta'(\chi(\cdot, 0) - \chi_s) = 0 \quad \text{a.e. in} \quad \Omega. \tag{5.33}$$

We multiply (5.32) by $\overline{\chi}^{Z\delta\varepsilon} - \chi$ and integrate in space. Since $\left(\overline{\xi}^{Z\delta\varepsilon} - \xi\right)\left(\overline{\chi}^{Z\delta\varepsilon} - \chi\right) \ge 0$ a.e. in Ω, by the monotonicity of β, (5.5), and (2.7), we find, using Green's formula, (5.6), (5.33) and Schwarz's inequality, that a.e. in $(0, T)$

$$\begin{aligned}
&\partial_t \frac{\delta'}{2}\left\|\widehat{\chi}^{Z\delta\varepsilon} - \chi\right\|^2 + \left\|\nabla\left(\sqrt{\varepsilon'}\left(\overline{\chi}^{Z\delta\varepsilon} - \chi\right)\right)\right\|^2 \\
&\le \left\|\delta'\left(\widehat{\chi}_t^{Z\delta\varepsilon} - \chi_t\right)\right\| \left\|\widehat{\chi}^{Z\delta\varepsilon} - \overline{\chi}^{Z\delta\varepsilon}\right\| - L\int_\Omega \left(\overline{u}^{Z\delta\varepsilon} - u\right)\left(\overline{\chi}^{Z\delta\varepsilon} - \chi\right)\,\mathrm{d}x \\
&\quad + (\delta - \delta')\left\|\widehat{\chi}_t^{Z\delta\varepsilon}\right\|\left\|\overline{\chi}^{Z\delta\varepsilon} - \chi\right\| + (\varepsilon - \varepsilon')\left\|\Delta\overline{\chi}^{Z\delta\varepsilon}\right\|\left\|\overline{\chi}^{Z\delta\varepsilon} - \chi\right\|.
\end{aligned} \tag{5.34}$$

We obtain from integration over t, applying (5.7), Hölder's inequality, (5.31), (5.33), (5.8), (5.30), and (5.13) that

$$\begin{aligned}
&\frac{\delta'}{2}\left\|\widehat{\chi}^{Z\delta\varepsilon}(s) - \chi(s)\right\|^2 + \int_0^s \left\|\nabla\left(\sqrt{\varepsilon'}\left(\overline{\chi}^{Z\delta\varepsilon}(t) - \chi(t)\right)\right)\right\|^2\,\mathrm{d}t \\
&\le \frac{\delta'}{2}\left\|\chi_{0\delta\varepsilon} - \chi_s\right\|^2 + T\left(\delta'\sqrt{\frac{C_1^*}{\delta}} + \|\delta'\chi_t\|_{L^\infty(0,T;L^2(\Omega))}\right)\sqrt{\frac{C_6^*}{\delta}}\,|Z| \\
&- L\int_0^s\int_\Omega \left(\overline{u}^{Z\delta\varepsilon} - u\right)\left(\overline{\chi}^{Z\delta\varepsilon} - \chi\right)\,\mathrm{d}x\,\mathrm{d}t + C_{15}^*\left(\frac{\delta - \delta'}{\sqrt{\delta}} + \frac{\varepsilon - \varepsilon'}{\sqrt{\varepsilon}}\right)\left\|\overline{\chi}^{Z\delta\varepsilon} - \chi\right\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned} \tag{5.35}$$

In the framework of Theorem 2.4 and Theorem 2.6, we have $\delta' = 0$, and in the framework of Theorem 2.3 and 2.5, we have $\delta' = \delta > 0$, $\chi_t \in L^\infty(0, T; L^2(\Omega))$, and $\chi_{0\delta\varepsilon} = \chi_s$. Hence, adding (5.29) and (5.35), applying (5.31), Gronwall's inequality, and (2.23), we find that

$$\begin{aligned}
&\max_{0\le s\le T}\left\|\nabla\int_0^s \left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\right\|^2 + \max_{0\le s\le T}\left\|\int_0^s \gamma(\tau)\left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\right\|_{L^2(\Gamma)}^2 \\
&+ \left\|\frac{\overline{u}^{Z\delta\varepsilon} - u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{\delta'}{2}\left\|\widehat{\chi}^{Z\delta\varepsilon} - \chi\right\|_{L^\infty(0,T;L^2(\Omega))}^2 + \left\|\nabla\left(\sqrt{\varepsilon'}\left(\overline{\chi}^{Z\delta\varepsilon} - \chi\right)\right)\right\|_{L^2(0,T;L^2(\Omega))}^2 \\
&\le C_{16}^*\Bigg(|Z| + \left(\frac{\delta - \delta'}{\sqrt{\delta}} + \frac{\varepsilon - \varepsilon'}{\sqrt{\varepsilon}}\right)\left\|\overline{\chi}^{Z\delta\varepsilon} - \chi\right\|_{L^2(0,T;L^2(\Omega))} \\
&\quad + \|e_{0\delta\varepsilon} - e_s\|\left\|\overline{u}^{Z\delta\varepsilon} - u\right\|_{L^2(0,T;L^2(\Omega))}\Bigg).
\end{aligned} \tag{5.36}$$

Next, we use calculations analogous to those used in Lemma 4.7 and Lemma 4.8 to improve the above estimate. Using Hölder's inequality, we obtain for $t \in (0,T)$

$$\left\|\overline{u}^{Z\delta\varepsilon}(t) - u(t)\right\|^2_{L^{\frac{3}{2}}(\Omega)} \;\leq\; \left\|\frac{\overline{u}^{Z\delta\varepsilon}(t) - u(t)}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|^2 \left\|\overline{u}^{Z\delta\varepsilon}(t)\right\|_{L^6(\Omega)} \|u(t)\|_{L^6(\Omega)} . \tag{5.37}$$

Therefore, by (A.1), (5.10), and (5.19),

$$\left\|\overline{u}^{Z\delta\varepsilon} - u\right\|^2_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} \leq C^*_{17} \left\|\frac{\overline{u}^{Z\delta\varepsilon} - u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|^2_{L^2(0,T;L^2(\Omega))} . \tag{5.38}$$

We have, by (5.5), (2.6), and Hölder's inquality, for $t \in (0,T)$

$$\left\|\overline{\theta}^{Z\delta\varepsilon}(t) - \theta(t)\right\|_{L^1(\Omega)} = \int\limits_\Omega \frac{\left|\overline{u}^{Z\delta\varepsilon}(t) - u(t)\right|}{\overline{u}^{Z\delta\varepsilon}(t)u(t)}\,\mathrm{d}x \;\leq\; \left\|\frac{\overline{u}^{Z\delta\varepsilon}(t) - u(t)}{\sqrt{\overline{u}^{Z\delta\varepsilon}(t)u(t)}}\right\| \left(\left\|\overline{\theta}^{Z\delta\varepsilon}(t)\right\| \|\theta(t)\|\right)^{\frac{1}{2}} .$$

Thus, (5.11) and (5.19) yield

$$\left\|\overline{\theta}^{Z\delta\varepsilon} - \theta\right\|^2_{L^2(0,T;L^1(\Omega))} \leq C^*_{18} \left\|\frac{\overline{u}^{Z\delta\varepsilon} - u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|^2_{L^2(0,T;L^2(\Omega))} . \tag{5.39}$$

Inserting $v = 1$ in (5.20), squaring, and using Schwarz's and Young's inequalities, we arrive for a.e. $t \in (0,T)$ at

$$\begin{aligned}
&\left| L \int\limits_\Omega \left(\widehat{\chi}^{Z\delta\varepsilon}(t) - \chi(t)\right)\,\mathrm{d}x \right|^2 \\
&\leq \;\; C^*_{19}\Bigg(\left\|\widehat{\theta}^{Z\delta\varepsilon}(t) - \theta(t)\right\|^2_{L^1(\Omega)} + \|e_{0\delta\varepsilon} - e_s\|^2 + \left\|\int\limits_0^t \gamma(\tau)\left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\right\|^2_{L^2(\Gamma)} \\
&\qquad + \left\|\gamma^Z - \gamma\right\|^2_{L^2(0,T;L^2(\Gamma))} \left\|\overline{u}^{Z\delta\varepsilon}(t)\right\|^2_{L^2(\Gamma)} + \left\|\zeta^Z - \zeta\right\|^2_{L^2(0,T;L^2(\Gamma))} + \left\|g^Z - g\right\|^2_{L^2(0,T;L^2(\Omega))}\Bigg).
\end{aligned}$$

Thus, by integration over t, Lemma 5.1, Schwarz's inequality, (5.10), (5.31), (5.13), and (5.15), we get

$$\begin{aligned}
\varepsilon \int\limits_0^T \left|\int\limits_\Omega \left(\overline{\chi}^{Z\delta\varepsilon}(t) - \chi(t)\right)\,\mathrm{d}x\right|^2 \mathrm{d}t \;\;\leq\;\; & C^*_{20}\Bigg(|Z|^2 + \left\|\overline{\theta}^{Z\delta\varepsilon} - \theta\right\|^2_{L^2(0,T;L^1(\Omega))} + \|e_{0\delta\varepsilon} - e_s\|^2 \\
& + \max_{0\leq t\leq T}\left\|\int\limits_0^t \gamma(\tau)\left(\overline{u}^{Z\delta\varepsilon}(\tau) - u(\tau)\right)\,\mathrm{d}\tau\right\|^2_{L^2(\Gamma)}\Bigg).
\end{aligned} \tag{5.40}$$

Recalling (5.36), (5.38), (5.39), (5.40), and Poincaré's inequality, we see that

$$\left\|\frac{\overline{u}^{Z\delta\varepsilon} - u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\overline{u}^{Z\delta\varepsilon} - u\right\|^2_{L^2(0,T;L^{\frac{3}{2}}(\Omega))} + \left\|\overline{\theta}^{Z\delta\varepsilon} - \theta\right\|^2_{L^2(0,T;L^1(\Omega))}$$

$$
\begin{aligned}
&+\frac{\delta'}{2}\left\|\widehat{\chi}^{Z\delta\varepsilon}-\chi\right\|^2_{L^\infty(0,T;L^2(\Omega))}+\left\|\sqrt{\varepsilon'}\left(\overline{\chi}^{Z\delta\varepsilon}-\chi\right)\right\|^2_{L^2(0,T;H^1(\Omega))}\\
\le\;& C^*_{21}\Bigg(|Z|+\left(\frac{\delta-\delta'}{\sqrt{\delta}}+\frac{\varepsilon-\varepsilon'}{\sqrt{\varepsilon}}\right)\left\|\overline{\chi}^{Z\delta\varepsilon}-\chi\right\|_{L^2(0,T;L^2(\Omega))}\\
&+\left\|e_{0\delta\varepsilon}-e_s\right\|\left\|\overline{u}^{Z\delta\varepsilon}-u\right\|_{L^2(0,T;L^2(\Omega))}+\left\|e_{0\delta\varepsilon}-e_s\right\|^2\Bigg). \qquad (5.41)
\end{aligned}
$$

By (5.41), (5.8), (5.10), (5.19), and (5.30), we obtain

$$
\begin{aligned}
&\left\|\overline{u}^{Z\delta\varepsilon}-u\right\|^2_{L^2(0,T;L^{\frac{3}{2}}(\Omega))}+\left\|\overline{\theta}^{Z\delta\varepsilon}-\theta\right\|^2_{L^2(0,T;L^1(\Omega))}\\
\le\;& C^*_{22}\left(\frac{\delta-\delta'}{\sqrt{\delta}}+\frac{\varepsilon-\varepsilon'}{\sqrt{\varepsilon}}+\left\|e_{0\delta\varepsilon}-e_s\right\|+\left\|e_{0\delta\varepsilon}-e_s\right\|^2\right). \qquad (5.42)
\end{aligned}
$$

We consider the convergences for $\delta\to\delta'$, $\varepsilon\to\varepsilon'$, and $|Z|\to 0$. Thus, (5.42) and (4.9) yield

$$
\begin{aligned}
\overline{u}^{Z\delta\varepsilon}\to u &\quad \text{strongly in} \quad L^2(0,T;L^{\frac{3}{2}}(\Omega)), \qquad (5.43)\\
\overline{\theta}^{Z\delta\varepsilon}\to \theta &\quad \text{strongly in} \quad L^2(0,T;L^1(\Omega)). \qquad (5.44)
\end{aligned}
$$

It follows from (5.10) that $\left(\overline{u}^{Z\delta\varepsilon}\right)$ and $\left(\widehat{u}^{Z\delta\varepsilon}\right)$ are uniformly bounded in $L^\infty(0,T;H^1(\Omega))$, that $\left(\widehat{u}^{Z\delta\varepsilon}\right)$ is uniformly bounded in $H^1(0,T;L^{\frac{3}{2}}(\Omega))$, and (5.11) yields that $\left(\overline{\theta}^{Z\delta\varepsilon}\right)$ and $\left(\widehat{\theta}^{Z\delta\varepsilon}\right)$ are uniformly bounded in $L^\infty(0,T;L^2(\Omega))$. Using compactness (see [Zei90a, Prob. 23.12]), (5.43), (5.14), (5.44), and (5.15), we see that the convergences (2.70) and (2.61) hold; moreover, recalling (5.20), (4.9), and Lemma 5.1, we have for all $v\in L^2(0,T;H^1(\Omega))$

$$
L\int_0^T\int_\Omega\left(\widehat{\chi}^{Z\delta\varepsilon}(t)-\chi(t)\right)v(t)\,\mathrm{d}x\;\mathrm{d}t\;\longrightarrow 0, \qquad (5.45)
$$

since $\gamma^Z v\longrightarrow\gamma v$ strongly in $L^2(0,T;L^2(\Gamma))$. Thus, we obtain (2.60) from (5.8) and compactness.

In the situation in Theorem 2.6, we have $\varepsilon'=\delta'=0$. Since we have, by (5.32) and (5.8),

$$
\left\|\overline{\xi}^{Z\delta\varepsilon}-\xi\right\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))}\le L\left\|\overline{u}^{Z\delta\varepsilon}-u\right\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega))}+C^*_{22}\left(\sqrt{\delta}+\sqrt{\varepsilon}\right),
$$

(2.68) follows using (5.42), (5.14), and (5.15). Moreover, (5.9) and compactness, yield that (2.69) hold. Since the convergences (2.60), (2.61), and (2.70) have already been shown, Theorem 2.6 is proved.

It remains to examine the passage to the limit in the Theorems 2.3, 2.4 and 2.5. We have $\min\{\varepsilon',\delta'\}>0$. Therefore, it follows from (5.10) that $\left(\overline{u}^{Z\delta\varepsilon}\right)$ and $\left(\widehat{u}^{Z\delta\varepsilon}\right)$ are uniformly bounded in $L^\infty(\Omega_T)$, and that $\left(\widehat{u}^{Z\delta\varepsilon}\right)$ is uniformly bounded in $H^1(0,T;L^2(\Omega))$. Using compactness, (2.70), and (5.14), we see that the convergences (2.50) and (2.51) hold. Thus, $u\in L^\infty(\Omega_T)$, and therefore, by (5.10) and (5.14),

$$
\left\|\widehat{u}^{Z\delta\varepsilon}-u\right\|_{L^2(0,T;L^2(\Omega))}+\left\|\overline{u}^{Z\delta\varepsilon}-u\right\|_{L^2(0,T;L^2(\Omega))}
$$

$$\leq \left\|\widehat{u}^{Z\delta\varepsilon}-\overline{u}^{Z\delta\varepsilon}\right\|_{L^2(0,T;L^2(\Omega))}+2\left(\left\|\overline{u}^{Z\delta\varepsilon}\right\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)}\right)^{\frac{1}{2}}\left\|\frac{\overline{u}^{Z\delta\varepsilon}-u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|_{L^2(0,T;L^2(\Omega))}$$

$$\leq C_{23}^*\left(|Z|+\left\|\frac{\overline{u}^{Z\delta\varepsilon}-u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|_{L^2(0,T;L^2(\Omega))}\right). \tag{5.46}$$

Hence, we obtain by applying (5.41), Young's inequality, (5.13), and (5.15), that

$$\begin{aligned}
&\left\|\frac{\overline{u}^{Z\delta\varepsilon}-u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|^2_{L^2(0,T;L^2(\Omega))}+\left\|\widehat{u}^{Z\delta\varepsilon}-u\right\|^2_{L^2(0,T;L^2(\Omega))}+\left\|\widehat{\theta}^{Z\delta\varepsilon}-\theta\right\|^2_{L^2(0,T;L^1(\Omega))}\\
&+\frac{\delta'}{4}\left\|\widehat{\chi}^{Z\delta\varepsilon}-\chi\right\|^2_{L^\infty(0,T;L^2(\Omega))}+\frac{1}{2}\left\|\sqrt{\varepsilon'}\left(\widehat{\chi}^{Z\delta\varepsilon}-\chi\right)\right\|^2_{L^2(0,T;H^1(\Omega))}\\
\leq\ & C_{24}^*\left(|Z|+\left(\frac{(\delta-\delta')^2}{\delta}+\frac{(\varepsilon-\varepsilon')^2}{\varepsilon}\right)\frac{1}{\max\{\delta',\varepsilon'\}}+\|e_{0\delta\varepsilon}-e_s\|^2\right).
\end{aligned} \tag{5.47}$$

In the situation of Theorem 2.4, we have $\delta'=\delta>0$, and $\varepsilon\longrightarrow\varepsilon'=0$. Applying (5.47), (4.7), (2.60), (5.8), (5.13), and compactness, we obtain that (2.58), (2.59), and

$$\varepsilon\overline{\chi}^{Z\delta\varepsilon}\to 0 \quad \text{strongly in} \quad L^2(0,T;H^2(\Omega)) \tag{5.48}$$

hold. Therefore, comparing the terms in (5.32), applying (2.59), (2.50), and (5.14), we have

$$\overline{\xi}^{Z\delta\varepsilon}\to\xi \qquad \text{weakly in} \quad L^2(0,T;L^2(\Omega)). \tag{5.49}$$

Thus, (5.9) and compactness lead to (2.49). Since we have already shown that (2.50), (2.51), (2.60), and (2.61) hold, Theorem 2.4 is proved.

It remains the passage to the limit in Theorems 2.3 and 2.5. We have $\varepsilon'>0$. Therefore, (5.11) leads to a uniform $L^\infty(\Omega_T)\cap L^\infty(0,T;H^1(\Omega))$–bound for $\left(\overline{\theta}^{Z\delta\varepsilon}\right)$ and $\left(\widehat{\theta}^{Z\delta\varepsilon}\right)$ and to a uniform $H^1(0,T;L^2(\Omega))$–bound for $\widehat{\theta}^{Z\delta\varepsilon}$. Thus, we have by compactness, (5.15), and (2.61), that $\theta\in L^\infty(\Omega_T)$, the convergences (2.53) and (2.54) hold, and that

$$\begin{aligned}
&\left\|\widehat{\theta}^{Z\delta\varepsilon}-\theta\right\|_{L^2(0,T;L^2(\Omega))}\\
\leq\ &\left\|\widehat{\theta}^{Z\delta\varepsilon}-\overline{\theta}^{Z\delta\varepsilon}\right\|_{L^2(0,T;L^2(\Omega))}+\left(\left\|\overline{\theta}^{Z\delta\varepsilon}\right\|_{L^\infty(\Omega)}\|\theta\|_{L^\infty(\Omega)}\right)^{\frac{1}{2}}\left\|\frac{\overline{u}^{Z\delta\varepsilon}-u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|_{L^2(0,T;L^2(\Omega))}\\
\leq\ & C_{25}^*\left(|Z|+\left\|\frac{\overline{u}^{Z\delta\varepsilon}-u}{\sqrt{\overline{u}^{Z\delta\varepsilon}u}}\right\|_{L^2(0,T;L^2(\Omega))}\right).
\end{aligned} \tag{5.50}$$

For a given $t_*\in(0,T)$ and admissible decompositions Z with $|Z|\leq t_*$, we have, by (5.10), a uniform $L^2(t_*,T;H^2(\Omega))$–bound for $\widehat{u}^{Z\delta\varepsilon}$. Thus, compactness and (2.50) yield (2.52). Using (2.60), (5.8), (5.13), and compactness arguments, we obtain that (2.64) and

$$\overline{\chi}^{Z\delta\varepsilon}\longrightarrow\chi \quad \text{weakly in} \quad L^2(0,T;H^2(\Omega)) \tag{5.51}$$

hold. From $\delta\longrightarrow\delta'$ and (2.64) it follows that

$$\delta\widehat{\chi}_t^{Z\delta\varepsilon}\longrightarrow\delta'\chi_t \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)).$$

Therefore, a comparison of the terms in (5.32) yields, by (5.51), (2.50), and (5.14), that (5.49) holds. Thus, compactness and (5.9) lead to (2.49).
By remark 4.1, we have $e_{0\delta\varepsilon} = e_s$. Thus we obtain (2.46), resp. (2.63), from (5.47) and (5.50). Since we have already proved that (2.49)–(2.54), (2.60) and (2.64) are satisfied, Theorem 2.5 is proved.
In the framework of Theorem 2.3, we have $\delta = \delta' > 0$. Hence (2.64), (5.8), and compactness lead to the additional convergence $\widehat{\chi}^{Z\delta\varepsilon} \to \chi$ weakly–star in $W^{1,\infty}(0,T;L^2(\Omega))$, and thus, by (2.64), the convergences (2.47) and (2.48) hold. Since (2.49)–(2.54) have already been shown, Theorem 2.3 is verified.

This finally ends the proof of the Theorems 2.3 to 2.6 $\square$

A Appendix

Since Ω is a subset of $\mathbb{R}^3$, we obtain from Sobolev's embedding theorem:

Lemma A.1 *There is a positive constant C, such that*

$$\|v\|_{L^6(\Omega)} \leq C\,\|v\|_{H^1(\Omega)} \quad \forall\, v \in H^1(\Omega). \tag{A.1}$$

The following result is well–known.

Lemma A.2 *There are two positive constants C, C', such that, for all $v \in H^1(\Omega)$,*

$$\|v\|^2_{H^1(\Omega)} \leq C\left(\|\nabla v\|^2 + \|v\|^2_{L^2(\Gamma)}\right) \leq C'\,\|v\|^2_{H^1(\Omega)}. \tag{A.2}$$

The following classical elliptic estimate can be found in [Ama93, Remark 9.3 d].

Lemma A.3 *There is a positive constant C, such that for all $v \in H^2(\Omega)$*

$$\|v\|^2_{H^2(\Omega)} \leq C\left(\|\Delta v\|^2 + \left\|\frac{\partial v}{\partial n}\right\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|v\|^2\right). \tag{A.3}$$

In particular, for all $v \in H^2(\Omega)$ with $\frac{\partial v}{\partial n} = 0$ a.e. on Γ, it holds

$$\|v\|^2_{H^2(\Omega)} \leq C\left(\|\Delta v\|^2 + \|v\|^2\right). \tag{A.4}$$

Elementary calculations lead to:

Lemma A.4 *For $n \in \mathbb{N}$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in \mathbb{R}$, it holds*

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{i} b_j = \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) - \sum_{j=1}^{n-1} b_{j+1} \sum_{i=1}^{j} a_i, \tag{A.5}$$

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{i} a_j = \frac{1}{2}\left(\sum_{i=1}^{n} a_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n} a_i^2. \tag{A.6}$$

By having a close look at the proof in [Lau93], one can extend the Lemma A.1 in [Lau93, Lau94] to the following result.

Lemma A.5 *Let $a > 1$, $b \geq 0$, $c \in \mathbb{R}$, and p_0 be given numbers such that $p_0 + \frac{c}{a-1} > 0$. We consider the sequence (p_n) of positive real numbers defined by $p_{n+1} = ap_n + c$ for all $n \in \mathbb{N}_0$. Then, $\lim_{n\to\infty} p_n = \infty$ holds and there is a positive constant C, such that, for every $C_0 \geq 1$, every $C_1 \geq 1$, and every sequence (α_n) of real positive numbers, satisfying*

$$\alpha_0 \leq C_1^{p_0}, \qquad \alpha_n \leq C_0 p_n^b \max\left\{C_1^{p_n}, \alpha_{n-1}^a\right\} \quad \forall\, n \in \mathbb{N}, \tag{A.7}$$

it holds

$$\limsup_{n\to\infty} \alpha_n^{\frac{1}{p_n}} \leq C C_0^{\frac{1}{(a-1)p_0+c}} C_1^{2^{\frac{p_0(a-1)+|c|}{p_0(a-1)+c}}}. \tag{A.8}$$

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References

[Ama93] Herbert Amann. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In H.-J. Schmeisser and H. Triebel, editors, *Function Spaces, Differential Operators and Nonlinear Analysis.* B.G. Teubner, 1993.

[Bar76] Viorel Barbu. *Nonlinear semigroups and differential equations in Banach spaces.* Noordhoff International Publishing, 1976.

[Bré71] Haim Brézis. Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations. In Eduardo H. Zarantonello, editor, *Contributions to Nonlinear Functional Analysis*, pages 101–155. Academic Press, London, 1971.

[Col95] Pierluigi Colli. Error estimates for nonlinear Stefan problems obtained as asymptotic limits of a Penrose–Fife model. Proceedings, ICIAM, 1995. To appear in ZAMP.

[CS94] Pierluigi Colli and Jürgen Sprekels. Stefan problems and the Penrose–Fife phase field model. Preprint 127, WIAS Berlin, 1994. Submitted.

[CS95] Pierluigi Colli and Jürgen Sprekels. On a Penrose–Fife model with zero interfacial energy leading to a phase–field system of relaxed Stefan type. *Annali di Matematica pura ed applicata*, CLXIX(IV):269–289, 1995.

[HS] Werner Horn and Jürgen Sprekels. A numerical method for a singular system of parabolic equations in two space dimensions. In preparation.

[HSZ93] Werner Horn, Jürgen Sprekels, and Songmu Zheng. Global exisitence of smooth solutions to the Penrose–Fife model for Ising ferromagnets. Report 431, DFG, 1993. To appear in Advances in Mathematical Sciences and Applications.

[Lau93] Philippe Laurençot. *Etude de quelques Problèmes aux Dérivées Partielles non Linéaires.* PhD thesis, Université de Franche–Comté, 1993.

[Lau94] Philippe Laurençot. Solutions to a Penrose–Fife model of phase–field type. *Journal of Mathematical Analysis and Applications*, 185:262–274, 1994.

[SZ93] Jürgen Sprekels and Songmu Zheng. Global smooth solutions to a thermodynamically consistent model of phase–field type in higher space dimensions. *Journal of Mathematical Analysis and Applications*, 176(1):200–223, 1993.

[Tib90] Dan Tiba. *Optimal Control of Nonsmooth Distributed Parameter Systems.* Springer, 1990.

[Zei90a] Eberhard Zeidler. *Nonlinear Functional Analysis and its Applications II/A: Linear Monotone Operators.* Springer, 1990.

[Zei90b] Eberhard Zeidler. *Nonlinear Functional Analysis and its Applications II/B: Linear Monotone Operators.* Springer, 1990.